Solutions for Practice Exam for Midterm II, part 2

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4 Problem 4 (parts (b) and (c))

(b) Outline the steps that you will take to find the volume of V in simple, short English sentences.

1. Sketch the region and determine an equation for each boundary of the region.
2. Choose a coordinate system, say u, v, w.
3. Represent each boundary equation in the chosen coordinate system.
4. Choose an order of integration, say $\iiint du\, dv\, dw$.
5. Represent the region using a hierarchical system(s) of inequalities of the form

$$\begin{aligned} w_1 &\le w \le w_2, \\ v_1(w) &\le v \ \le v_2(w), \\ u_1(v,w) &\le u \ \le u_2(v,w). \end{aligned}$$

   (This in general requires iteratively projecting the region of integration and finding the boundary.)

6. Write down the Jacobian determinant $\frac{\partial x \partial y \partial z}{\partial u \partial v \partial w}$.

So the iterated integral is

$$\int_{w=w_1}^{w_2} \int_{v=v_1(w)}^{v_2(w)} \int_{u=u_1(v,w)}^{u_2(v,w)} \frac{\partial x \partial y \partial z}{\partial u \partial v \partial w}\, du\, dv\, dw.$$

(c) Outline the steps that you will take to find the mass of a substance contained in V with density given by the function $f(x, y, z)$ in simple, short English sentences.

Same procedure as before except that in addition you need to:

- Express the density function in terms of the variables you chose and include it in the integrand.

So the iterated integral is

$$\int_{w_1}^{w_2} \int_{v_1(w)}^{v_2(w)} \int_{u_1(v,w)}^{u_2(v,w)} \widetilde{f}(u, v, w) \frac{\partial x \partial y \partial z}{\partial u \partial v \partial w}\, du\, dv\, dw,$$

where

$$\widetilde{f}(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w)).$$

5 Problem 5

Use the setup and the notation from Problem (4). You are NOT required to calculate the numerical integration answers to the following; rather, you must show the steps in your work carefully and do all the required computations to write the multiple integrals as iterated integrals with explicit bounds.
(i) Write down the region of integration for calculation of the volume in spherical coordinates, then set up the triple integral and its bounds.

The region is:

$$\begin{aligned} 0 &\le \theta \le 2\pi, \\ \pi/4 &\le \phi \le \pi/2, \\ \sqrt(2) &\le \rho \le 2. \end{aligned}$$

The integral is:

$$\int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=\sqrt{2}}^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

(ii) Repeat part (i) above using cylindrical coordinates.

Solution:

$$\int_{\theta=0}^{2\pi} d\theta \int_{r=1}^{\sqrt{2}} r\, dr \int_{z=\sqrt{2-r^2}}^{r} dz + \int_{\theta=0}^{2\pi} d\theta \int_{r=\sqrt{2}}^{2} r\, dr \int_{z=0}^{\sqrt{4-r^2}} dz$$

or

$$\int_{\theta=0}^{2\pi} d\theta \int_{z=0}^{1} dz \int_{r=\sqrt{2-z^2}}^{\sqrt{4-z^2}} r\, dr + \int_{\theta=0}^{2\pi} d\theta \int_{z=1}^{\sqrt{2}} dz \int_{r=z}^{\sqrt{4-z^2}} r\, dr$$

(iii) Repeat part (i) above using Cartesian coordinates.

When we go from polar to cartesian coordinates, annulus integrals of the form

$$\int_{\theta=0}^{2\pi} d\theta \int_{r=r_1}^{r_2} r\, dr$$

become

$$\cancel{\int_{x=0}^{r_2} dx \int_{y=\sqrt{r_1^2-x^2}}^{\sqrt{r_2^2-x^2}} dy}$$

$$4\int_{x=0}^{r_1} dx \int_{y=\sqrt{r_1^2-x^2}}^{\sqrt{r_2^2-x^2}} dy + 4\int_{x=r_1}^{r_2} dx \int_{y=0}^{\sqrt{r_2^2-x^2}} dy.$$

So in cartesian coordinates the integral is:

$$\cancel{\int_{x=0}^{\sqrt{2}} dx \int_{y=\sqrt{1-x^2}}^{\sqrt{2-x^2}} dy \int_{z=\sqrt{2-r^2}}^{r} dz}$$

$$+\cancel{\int_{x=0}^{2} dx \int_{y=\sqrt{2-x^2}}^{\sqrt{4-x^2}} dy \int_{z=0}^{\sqrt{4-r^2}} dz}$$

or

$$\cancel{\int_{z=0}^{1} dz \int_{x=0}^{\sqrt{4-z^2}} dx \int_{y=\sqrt{2-z^2-x^2}}^{\sqrt{4-z^2-x^2}} dy.}$$

$$+\cancel{\int_{z=1}^{\sqrt{2}} dz \int_{x=0}^{\sqrt{4-z^2}} dx \int_{y=\sqrt{z^2-x^2}}^{\sqrt{4-z^2-x^2}} dy.}$$

So each cylindrical coordinates integral becomes in Cartesion coordinates a sum of two integrals.

6 Problem 6. Lagrange Multiplier.

Find all points $P = (x, y, z)$ at which the function $f(x, y, z) = 2x+3y+z+5$ attains a minimum subject to the constraint $g(x, y, z) = 4x^2 + 9y^2 - z = 0$.

$$\nabla f = \lambda \nabla g$$

says

$$\begin{aligned} 2 &= 8x\lambda, \\ 3 &= 18y\lambda, \\ 1 &= -\lambda; \end{aligned}$$

i.e.,

$$\begin{aligned} 1 &= -4x, \\ 1 &= -6y. \end{aligned}$$

So solving and invoking the constraint,

$$x = -\frac{1}{4}, y = -\frac{1}{6}, z = \frac{1}{2}.$$

At this lone critical point $f(-\frac{1}{4}, -\frac{1}{6}, \frac{1}{2}) = \frac{9}{2}$. This is the only candidate for a minimum. To see that a minimum must exist (and to confirm the minimizer) eliminate the constraint and complete the square.