p962 #34. Find parametric equations for the line tangent to the helix

$$\vec{r}(t) = \begin{pmatrix} \sqrt{2}\cos t \\ \sqrt{2}\sin t \\ t \end{pmatrix} \text{ at } t = \frac{\pi}{4}$$

Solution:

$$\vec{r}(t) \simeq \underbrace{\vec{r}(t_0) + \vec{r}\prime(t_0)(t - t_0)}_{\substack{\text{general formula} \\ \text{for tangent line} \\ \text{at } t_0 \text{ in paramete } t.}} =: L$$

$$\vec{r}\prime(t) = \begin{pmatrix} -\sqrt{2}\sin t \\ \sqrt{2}\cos t \\ 1 \end{pmatrix}$$

$$\vec{r}\prime\left(\frac{\pi}{4}\right) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \qquad \vec{r}\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1 \\ 1 \\ \frac{\pi}{4} \end{pmatrix}$$

So the tangent line has parametrization

$$L = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \left(t - \frac{\pi}{4}\right)\begin{pmatrix} 1 \\ 1 \\ \frac{\pi}{4} \end{pmatrix}.$$

Change the parametrization: let $\tilde{t} = t - \frac{\pi}{4}$. Then

$$L = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \tilde{t}\begin{pmatrix} 1 \\ 1 \\ \frac{\pi}{4} \end{pmatrix}, \text{ i.e.}$$

$$\begin{aligned} x(\tilde{t}) &= -1 + \tilde{t}, \\ y(\tilde{t}) &= 1 + \tilde{t}, \\ z(\tilde{t}) &= 1 + \tilde{t}\frac{\pi}{4}. \end{aligned}$$

p963 #3. A frictionless particle P starts from rest at time $t = 0$ at point $(a, 0, 0)$, and slides down the helix

$$\vec{r}(\theta) = \begin{pmatrix} a\cos\theta \\ a\sin\theta \\ b\theta \end{pmatrix}, \ (a, b > 0).$$

Conservation says that the speed after falling a distance z $(= b\theta)$ is $\sqrt{2gz}$. where g is the gravitational acceleration.

(a) Find the angular velocity $\frac{d\theta}{dt}$ when $\theta = 2\pi$.

Solution: the speed of the particle is

$$\left\| \frac{d}{dt}\vec{r}(\theta(t)) \right\|.$$

But by the chain rule,

$$\frac{d}{dt}\vec{r}(\theta(t)) = \vec{r}\prime(\theta(t))\frac{d\theta}{dt}.$$

Now

$$\vec{r}\prime(\theta) = \begin{pmatrix} -a\sin\theta \\ a\cos\theta \\ b \end{pmatrix}.$$

The height of the particle is $z = b\theta$, so the speed is $\sqrt{2gb\theta}$. So $\sqrt{2gb\theta} = \|\vec{r}\prime(\theta)\|\frac{d\theta}{dt}$. But $\|\vec{r}\prime(\theta)\|^2 = a^2 + b^2$. So

$$\frac{d\theta}{dt} = \sqrt{\frac{2gb\theta}{a^2 + b^2}} \qquad (*)$$

So

$$\left.\frac{d\theta}{dt}\right|_{\theta=2\pi} = 2\sqrt{\frac{gb\pi}{a^2 + b^2}}.$$

(b) Find $\theta(t)$ and $z(t)$.

Solution: By $(*)$ we have

$$\frac{dt}{d\theta} = \sqrt{\frac{a^2 + b^2}{2gb\theta}} = \left(\sqrt{\frac{a^2 + b^2}{2gb}}\right)\theta^{-\frac{1}{2}},$$

Integrating, we get

$$t = 2\left(\sqrt{\frac{a^2 + b^2}{2gb}}\right)\theta^{\frac{1}{2}} + C.$$

Because the particle is at $(a, 0, 0)$ when $t = 0$, we conclude that $C = 0$. So

$$\theta = \left(\frac{gb}{2}\right)\left(\frac{1}{a^2+b^2}\right)t^2,$$
$$z = b\theta = \left(\frac{gb^2}{2}\right)\left(\frac{1}{a^2+b^2}\right)t^2.$$

Remark:

$$\frac{d\theta}{dt} = \frac{gb}{a^2+b^2}t.$$

(c) Find the tangential and normal components of the velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$. Does the acceleration have any nonzero component in the direction of the binormal vector $\mathbf{B}$?

Solution: Using the formula for $\theta(t)$ from part (b), the particle's speed is

$$\|\vec{v}(t)\| = \sqrt{2gb\theta(t)} = \frac{gbt}{\sqrt{a^2+b^2}}.$$

By definition, $\mathbf{T} = \frac{\vec{v}}{\|\vec{v}|}$, so

$$\vec{v}(t) = \|\vec{v}(t)\|\mathbf{T} = \frac{gbt}{\sqrt{a^2+b^2}}\mathbf{T}.$$

The tangent vector is independent of the parametrization, so we can use θ and get,

$$\mathbf{T} = \begin{pmatrix} -a\sin\theta \\ a\cos\theta \\ b \end{pmatrix} \frac{1}{\sqrt{a^2+b^2}}$$

So the normal vector is

$$\mathbf{N} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$$

The acceleration is

$$
\begin{aligned}
\vec{a}(t) =& \frac{d^2\vec{r}}{dt^2} \\
=& \frac{d}{dt}\begin{pmatrix} -a\sin\theta \\ a\cos\theta \\ b \end{pmatrix}\frac{d\theta}{dt} \\
=& \begin{pmatrix} -a\cos\theta \\ -a\sin\theta \\ 0 \end{pmatrix}\left(\frac{d\theta}{dt}\right)^2 + \begin{pmatrix} -a\sin\theta \\ a\cos\theta \\ b \end{pmatrix}\frac{d^2\theta}{dt^2} \\
=& \begin{pmatrix} -a\cos\theta \\ -a\sin\theta \\ 0 \end{pmatrix}\left(\frac{gbt}{a^2+b^2}\right)^2 + \begin{pmatrix} -a\sin\theta \\ a\cos\theta \\ b \end{pmatrix}\frac{gb}{a^2+b^2} \\
=& \frac{gb}{\sqrt{a^2+b^2}}\mathbf{T} + a\left(\frac{gbt}{a^2+b^2}\right)^2\mathbf{N}.
\end{aligned}
$$

Because it is a linear combination of $\mathbf{T}$ and $\mathbf{N}$ there is no component in the direction of $\mathbf{B}$.

p964 #11. When the position of a particle moving in space is given in cylindrical coordinates, the unit vectors we use to describe its position are

$$\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}, \qquad \mathbf{u}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j},$$

and $\mathbf{k}$. The particle's position vector is then $\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$, where r is the positive polar distance coordinate of the particle's position.

(a) Show that $\mathbf{u}_r$, $\mathbf{u}_\theta$ and $\mathbf{k}$ for a right-handed frame of unit vectors.

Solution: They are unit vectors because

$$
\begin{aligned}
\mathbf{u}_r \cdot \mathbf{u}_r =& \cos^2\theta + \sin^2\theta = 1, \\
\mathbf{u}_\theta \cdot \mathbf{u}_\theta =& \sin^2\theta + \cos^2\theta = 1, \\
\mathbf{k} \cdot \mathbf{k} =& 1^2 = 1.
\end{aligned}
$$

They are mutually orthogonal because

$$
\begin{aligned}
\mathbf{u}_r \cdot \mathbf{u}_\theta =& -\cos\theta\sin\theta + \sin\theta\cos\theta = 0, \\
\mathbf{u}_\theta \cdot \mathbf{k} =& -0\sin\theta + 0\cos\theta + 0(1) = 0, \\
\mathbf{k} \cdot \mathbf{u}_r =& 0\cos\theta + 0\sin\theta + 0(1) = 0.
\end{aligned}
$$

All that remains is to show that $\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k}$, but

$$\mathbf{u}_r \times \mathbf{u}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \cos^2\theta\mathbf{k} - (-\sin^2\theta\mathbf{k}) = \mathbf{k}.$$

(b) Show that

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta \text{ and } \frac{d\mathbf{u}_{theta}}{d\theta} = -\mathbf{u}_r$$

Solution:

$$\frac{d\mathbf{u}_r}{d\theta} = \frac{d}{d\theta}(\cos\theta)\mathbf{i} + \frac{d}{d\theta}(\sin\theta)\mathbf{j} = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta$$
$$\frac{d\mathbf{u}_\theta}{d\theta} = -\frac{d}{d\theta}(\sin\theta)\mathbf{i} + \frac{d}{d\theta}(\cos\theta)\mathbf{j} = (-\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\,\mathbf{u}_r$$

(c) Assuming that the necessary derivatives with respect to t exist, express $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \ddot{\mathbf{r}}$ in terms of $\mathbf{u}_r$, $\mathbf{u}_\theta$, $\mathbf{k}$, $\dot{r}$, $\dot{\theta}$, $\dot{z}$, $\ddot{r}$, $\ddot{\theta}$, $\ddot{z}$.

Solution: We are given $\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$, where $\mathbf{r}$ and $\mathbf{u}_r$ are vectors that depend on t, r and z are scalars that depend on t and $\mathbf{k}$ is a constant vector. Therefore

$$\begin{aligned}
\mathbf{v} = \dot{\mathbf{r}} =& \frac{d}{dt}(r\mathbf{u}_r + z\mathbf{k}) \\
=& \frac{d}{dt}r\mathbf{u}_r + \frac{d}{dt}z\mathbf{k} \\
=& \dot{r}\mathbf{u}_r + r\frac{\mathbf{u}_r}{dt} + \dot{z}\mathbf{k} \\
=& \dot{r}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} + \dot{z}\mathbf{k} \\
=& \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}
\end{aligned}$$

For the acceleration we get

$$\begin{aligned}
\mathbf{a} = \dot{\mathbf{r}} =& \frac{d}{dt}(\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}) \\
=& \frac{d}{dt}\dot{r}\mathbf{u}_r + \frac{d}{dt}r\dot{\theta}\mathbf{u}_\theta + \frac{d}{dt}\dot{z}\mathbf{k} \\
=& \ddot{r}\mathbf{u}_r + \dot{r}\frac{d\mathbf{u}_r}{dt} + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\frac{d\mathbf{u}_\theta}{dt} + \ddot{z}\mathbf{k} \\
=& \ddot{r}\mathbf{u}_r + \dot{r}\frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\frac{d\mathbf{u}_\theta}{d\theta}\frac{d\theta}{dt} + \ddot{z}\mathbf{k} \\
=& \ddot{r}\mathbf{u}_r + \dot{r}\mathbf{u}_\theta\dot{\theta} + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta - r\dot{\theta}\mathbf{u}_r\dot{\theta} + \ddot{z}\mathbf{k} \\
=& (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\ddot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}
\end{aligned}$$

p964 #12. (a) Show that when you express $ds^2 = dx^2 + dy^2 + dz^2$ in terms of cylindrical coordinates, you get $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.

Solution: Expressing cartesian coordinates in terms of cylindrical coordinates we get

$$\begin{aligned} x =& r\cos\theta \\ y =& r\sin\theta \\ z =& z. \end{aligned}$$

Differentiating we get

$$\begin{aligned} dx =& \cos\theta dr - r\sin\theta d\theta \\ dy =& \sin\theta dr + r\cos\theta d\theta \\ dz =& dz. \end{aligned}$$

Squaring we get

$$\begin{aligned} dx^2 =& \cos^2\theta dr^2 - 2r\sin\theta\cos\theta dr d\theta + r^2\sin^2\theta d\theta^2 \\ dy^2 =& \sin^2\theta dr^2 + 2r\sin\theta\cos\theta dr d\theta + r^2\cos^2\theta d\theta^2 \\ dz^2 =& dz^2. \end{aligned}$$

Adding we get

$$ds^2 = dx^2 + dy^2 + dz^2 = ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

(b) Interpret this result geometrically in terms of the edges and a diagonal of a box. Sketch the box.

Solution: Insert solution here.

(c) Use the result in part (a) to find the length of the curve $r = e^\theta$, $z = e^\theta$, $0 \le \theta \le \ln 8$.

Solution: Because $r = z = e^\theta$ we get $dr = dz = e^\theta d\theta$, so

$$\begin{aligned} L =& \int_0^{\ln 8} \sqrt{dr^2 + r^2 d\theta^2 + dz^2} \\ =& \int_0^{\ln 8} \sqrt{e^{2\theta} d\theta^2 + e^{2\theta} d\theta^2 + e^{2\theta} d\theta^2} \\ =& \int_0^{\ln 8} \sqrt{3e^{2\theta} d\theta^2} \\ =& \int_0^{\ln 8} \sqrt{3}e^\theta d\theta = \left[\sqrt{3}e^\theta\right]_0^{\ln 8} \\ =& 8\sqrt{3} - \sqrt{3} = 7\sqrt{3}. \end{aligned}$$