Bickler, Susannah

1.6 #6, 10, 18, 22, 34, 36, 37

17/17

(6) $(x+2y)y' = y$

$$y' = \frac{y}{x+2y}$$

$$dy/dx = \frac{y/x}{1+2y/x} \quad \text{Let } v = y/x \Rightarrow y = xv$$

$$dy/dx = v + xv'$$

$$v + xv' = \frac{v}{1+2v}$$

$$xv' = \left(\frac{v}{1+2v} - v\right)$$

$$\int\left(\frac{1+2v}{2v^2}\right)dv = \int\frac{1}{x}dx$$

$$\frac{1}{2}\left(-\frac{x}{y} + 2\ln\frac{y}{x}\right) = \ln|x| = 2y\ln y = x + cy$$

(10) $$\frac{xyy'}{x^2} = \frac{x^2+3y^2}{x^2}$$

$$\frac{y}{x}y' = 1 + 3\left(\frac{y}{x}\right)^2 \quad \text{Let } v = \frac{y}{x}$$

$$y' = xv' + v$$

$$v(v + xv') = 1 + 3v^2$$

$$v + xv' = \frac{1}{v} + 3v$$

$$xv' = \frac{1}{v} + 2v$$

$$\int \frac{v}{1+2v^2}dv = \int\frac{1}{x}dx \quad \text{Let } u = 2v^2 + 1$$

$$du = 4v\,dv$$

$$\int \frac{\cancel{v}\,du}{u\,4\cancel{v}} = \ln|x| + C$$

$$\frac{1}{4}\int\frac{1}{u}du = \ln|x| + C$$

$$\frac{1}{4}\ln|u| = \ln|x| + C$$

$$\frac{1}{4}\ln|2v^2+1| = \ln|x| + C$$

$$\frac{1}{4}\ln|2(\tfrac{y}{x})^2+1| = \ln|x| + C$$

$$e^{(\ln|2(\frac{y}{x})^2+1|)} = e^{(4\ln|x|^4 + C')}$$

$$2\left(\frac{y}{x}\right)^2 + 1 = C''x^4$$

$$2\left(\frac{y}{x}\right)^2 = C''x^4 - 1$$

$$\frac{y^2}{x^2} = \frac{1}{2}C''x^4 - \frac{1}{2}$$

$$(y^2 = \tfrac{1}{2}C''x^6 - \tfrac{1}{2}x^2)2$$

$$2y^2 = C''x^6 - x^2$$

$$2y^2 + x^2 = C''x^6$$

(18) $(x+y)y' = 1$ Let $v = x+y$

$v(v'-1) = 1$ $v' = 1 + y'$

$(v'-1) = \frac{1}{v}$ $v'-1 = y'$

$\frac{dv}{dx} - 1 = \frac{1}{v}$

$\frac{dv}{dx} = \frac{1}{v} + 1$

$\frac{dv}{dx} = \frac{1+v}{v}$

$\int \frac{v\,dv}{1+v} = \int dx$ let $u = 1+v \rightarrow v = u-1$

$du = dv$

$du = dv$

$\int \frac{u-1}{u} du = \int dx$

$\int 1 - \frac{1}{u} du = \int dx$

$u - \ln|u| = x + C$

$1+v - \ln|1+v| = x + C$

$1 + \cancel{x} + y - \ln|1+x+y| = \cancel{x} + C$

$y = \ln|1+x+y| + (C-1)$

$\boxed{y = \ln|1+x+y| + C'}$

(22) $\dfrac{x^2y' + 2xy = 5y^4}{x^2 \qquad\qquad x^2}$ Let $v = y^{1-4} = y^{-3} \rightarrow y = v^{-1/3}$

$v' = dv/dx = -3y^{-4}\, y'$

$-\frac{1}{3}y^4 v' = y'$

$y' + \frac{2}{x}y = \frac{5}{x^2}y^4$

$-\frac{1}{3}y^4 v' + \frac{2}{x}v^{-1/3} = \frac{5}{x^2}v^{-4/3}$

$\left(-\frac{1}{3}y^4\frac{dv}{dx} + \frac{2}{x}\frac{1}{v^{1/3}} = \frac{5}{x^2}v^{-4/3}\right)3$

$(v^{-1/3})^4\, dv/dx - 6x^{-1}v^{-1/3} = -\frac{15}{x^2}v^{-4/3}) v^{+3/4}$

$\frac{dv}{dx} - 6x^{-1}v = -15x^{-2}$ $\qquad \rho = e^{-\int \frac{6}{x}dx} = e^{\ln x^{-6}} = x^{-6}$

$x^{-6}(v' - 6x^{-1}v) = x^{-6}(-15x^{-2})$

$\int \frac{d}{dx}(x^{-6}v) = -\int 15x^{-8}\,dx$

$x^{-6}v = \frac{+15}{+7x^7} + C$

$v = \left(\frac{15 + 7Cx^7}{7x^7}\right)x^6$

$v = \frac{15 + 7Cx^7}{7x}$

$y^{-3} = \frac{15 + 7Cx^7}{7x}$

$y^3 = \frac{7x}{(15 + 7Cx^7)}$

(34) $(2xy^2 + 3x^2)dx + (2x^2y + 4y^3)dy = 0$

Let $M = 2xy^2 + 3x^2\,dx$, $N = 2x^2y + 4y^3\,dy$

$\left.\begin{array}{l} \frac{\partial M}{\partial y} = N \Rightarrow \frac{\partial M}{\partial y} = 4xy \checkmark \\ \frac{\partial N}{\partial x} = M \Rightarrow \frac{\partial N}{\partial x} = 4xy \checkmark \end{array}\right\}$ equation is exact

$\int \frac{\partial F}{\partial x} = M = \int 2xy^2 + 3x^2\,dx$ $\qquad \boxed{F = x^2y^2 + x^3 + y^4 + C}$

$F = x^2y^2 + x^3 + g(y)$

$\frac{\partial F}{\partial y} = N \Rightarrow \frac{\partial F}{\partial y} = \cancel{2x^2y} + \frac{dg}{} = \cancel{2x^2y} + 4y^3$

$\int dg = \int 4y^3$

$g(y) = y^4 + C$

(36) $\underbrace{(1+ye^{xy})}_{M}dx + \underbrace{(2y+xe^{xy})}_{N}dy = 0$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = N \Rightarrow \dfrac{\partial M}{\partial y} = 2ye^{xy} + 2y \ \checkmark \\[2ex] \dfrac{\partial N}{\partial x} = M \Rightarrow \dfrac{\partial N}{\partial x} = 1 + ye^{xy} \ \checkmark \end{array}\right\} \text{equation is exact}$$

$$\frac{\partial F}{\partial x} = M \Rightarrow F = \int(1+ye^{xy})\,dx = x + e^{xy} + g(y)$$

$$\frac{\partial F}{\partial y} = N \Rightarrow \frac{\partial F}{\partial y} = \cancel{xe^{xy}} + dg = \cancel{xe^{xy}} + 2y$$

$$\int dg = \int 2y\,dy$$

$$g(y) = y^2 + C$$

$$F = x + e^{xy} + y^2 + C$$

(37) $\underbrace{(\cos x + \ln y)}_{M}dx + \underbrace{\left(\frac{x}{y} + e^y\right)}_{N}dy = 0$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = N \Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{1}{y} \ \checkmark \\[2ex] \dfrac{\partial N}{\partial x} = M \Rightarrow \dfrac{\partial N}{\partial x} = \dfrac{1}{y} \ \checkmark \end{array}\right\} \text{equation is exact}$$

$$\frac{\partial F}{\partial x} = M \Rightarrow \int \frac{\partial F}{\partial x} = \int(\cos x + \ln y)\,dx$$

$$F = \sin x + x\ln y + g(y)$$

$$\frac{\partial F}{\partial y} = N \Rightarrow \frac{\partial F}{\partial y} = \cancel{\frac{x}{y}} + \frac{dg}{dy} = \cancel{\frac{x}{y}} + e^y$$

$$\int dg = \int e^y\,dy$$

$$g(y) = e^y + C$$

$$F = \sin x + x\ln y + e^y + C$$

2.1 # 5, 6, 21, 28

(5) $\frac{dx}{dt} = 3x(5-x), \quad x(0) = 8$

$$\int \frac{dx}{3x(5-x)} = \int 1\,dt$$

$$-\frac{1}{15}\ln\left|\frac{5-x}{x}\right| = t + C$$

$$x(0) = 8$$

$$-\frac{1}{15}\ln\left|\frac{5-8}{8}\right| = 0 + C$$

$$-\frac{1}{15}\ln\left(\frac{3}{8}\right) = C$$

$$-\frac{1}{15}\ln\left|\frac{5-x}{x}\right| = t - \frac{1}{15}\ln\left(\frac{3}{8}\right)$$

$$e^{\left(\ln\left|\frac{5-x}{x}\right|\right)} = e^{\left(-15t + \ln\left(\frac{3}{8}\right)\right)}$$

$$\left|\frac{5-x}{x}\right| = e^{-15t} + \frac{3}{8}$$

$$5 - x = xe^{-15t} + \frac{3}{8}x$$

$$5 = xe^{-15t} + \frac{11}{8}x$$

$$5 = \left(e^{-15t} + \frac{11}{8}\right)x$$

$$x(t) = \left(\frac{5}{e^{-15t} + \frac{11}{8}}\right)\frac{8}{8}$$

$$\boxed{x(t) = \frac{40}{8 - 3e^{-15t}}}$$

$$\int \frac{dx}{3x(5-x)} = \int \frac{A}{3x}\,dx + \int \frac{B}{5-x}\,dx$$

$$\frac{A(5-x) + B(3x)}{3x(5-x)} = \frac{1}{3x(5-x)}$$

$$5A + (3B - A)x = 1$$

$$3B - A = 0, \quad 5A = 1$$

$$A = \frac{1}{5}$$

$$3B - \frac{1}{5} = 0$$

$$3B = \frac{1}{5}$$

$$B = \frac{1}{15}$$

so $\int \frac{A}{3x}\,dx + \int \frac{B}{5-x}\,dx$

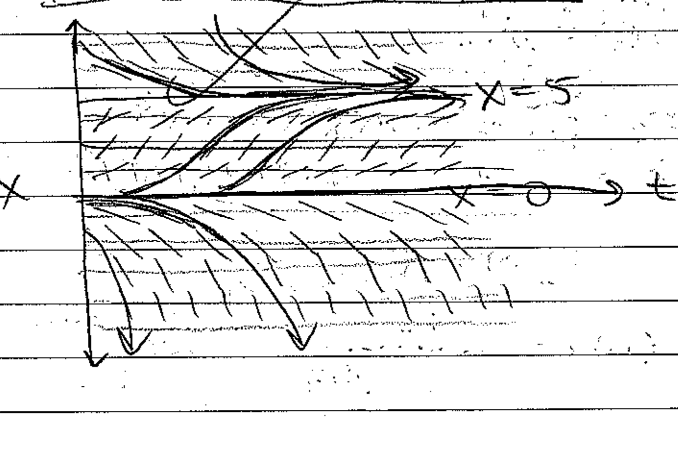
$$= \frac{1}{5}\int \frac{1}{(3x)}\,dx + \frac{1}{15}\int \frac{1}{(5-x)}\,dx$$

$$= \frac{1}{15}\ln|x| - \frac{1}{15}\ln|5-x|$$

$$= -\frac{1}{15}\left(\ln|x| + \ln|5-x|\right)$$

$$= -\frac{1}{15}\ln\left(\left|\frac{5-x}{x}\right|\right)$$

solutions: $x = 5$, $x = 0$



⑥ $\frac{dx}{dt} = 3x(x-5)$, $x(0) = 2$

$$\int \frac{dx}{3x(x-5)} = \int 1\, dt \rightarrow \frac{1}{3x(x-5)} = \frac{A}{3x} + \frac{B}{(x-5)} = \frac{Ax-5A+3xB}{3x(x-5)}$$

$$= \frac{(A+3B)x - 5A}{3x(x-5)} = \frac{1}{3x(x-5)}$$

$A+3B=0$ $\quad -5A=1$

$A = -1/5$

$-1/5 + 3B = 0$

$3B = 1/5$

$B = 1/15$

so $\frac{1}{3x(x-5)} = -\frac{1}{5(3x)} + \frac{1}{15(x-5)}$

$$-\frac{1}{15}\int \frac{1}{x}dx + \frac{1}{15}\int \frac{1}{x-5} = \int 1 dt$$

$$-\frac{1}{15}\ln|x| + \frac{1}{15}\ln|x-5| = t + C$$

$$+\frac{1}{15}\ln\left|\frac{x-5}{x}\right| = t + C$$

$x(0)=2$:

$$\frac{1}{15}\ln\left|\frac{2-5}{2}\right| = 0 + C$$

$$\frac{1}{15}\ln\left(\frac{3}{2}\right) = C$$

so $-\frac{1}{15}\ln\left|\frac{x-5}{x}\right| = t + \frac{1}{15}\ln\left(\frac{3}{2}\right)$

$$e^{\left(\ln\left|\frac{x-5}{x}\right|\right)} = e^{\left(15t + \ln\left(\frac{3}{2}\right)\right)}$$

$$\left|\frac{x-5}{x}\right| = e^{15t} e^{\ln\left(\frac{3}{2}\right)}$$

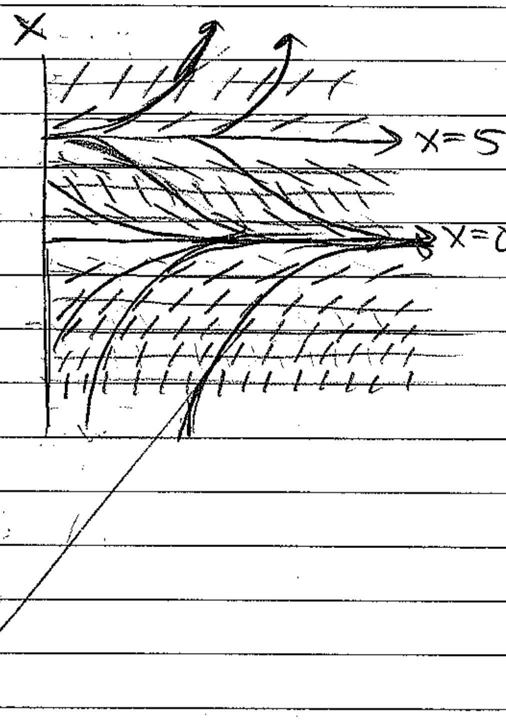
$$\frac{x-5}{x} = \frac{3}{2}e^{15t}$$

$$x-5 = \frac{3}{2}e^{15t}x$$

$$\left(1+\frac{3}{2}e^{15t}\right)x = 5$$

$$x = \left(\frac{5}{1+\frac{3}{2}e^{15t}}\right)\frac{2}{2}$$

$$X(t) = \frac{10}{2+3e^{15t}}$$



solutions for $x(t)$: $x=5$, $x=0$

(21) $\frac{dP}{dt} = kP(200-P)$, Let $x=0$ be 1940 , (0,100)

$1 = k(101)(200-101)$ and y be in millions (60 , ?)

$k = 0.0001$

$\frac{dP}{dt} = kP(200-P)$

$dP = kP(200-P)dt$

$\frac{dP}{kP(200-P)} = dt$

$\frac{1}{kP(200-P)} = \frac{A}{kP} + \frac{B}{(200-P)}$, $k = .0001$

$= \frac{200A - AP + kBP}{kP(200-P)}$

$= \frac{(kB-A)P + 200A}{kP(200-P)}$

$50(.0001)B - A = 0$, $200A = 1$

$.0001B = 1/200$, $A = \frac{1}{200}$

$B = 50$

$\boxed{\frac{1}{kP(200-P)} = \frac{1}{200}\left(\frac{1}{kP}\right) + \frac{50}{(200-P)}}$

$50\int \frac{1}{P}dP + 50\int \frac{1}{200-P}dP = \int dt$

$50\ln|P| - 50\ln|200-P| = t + C$

$50\left(\ln\left|\frac{P}{200-P}\right|\right) = t + C$

$e^{\left(\ln\left|\frac{P}{200-P}\right|\right)} = e^{(t/50 + C')}$

$\frac{P}{200-P} = C''e^{t/50}$

$P = (C''e^{t/50})(200-P)$

$C'''e^{-t/50}P = 200 - P$

$(C'''e^{-t/50} + 1)P = 200$

$P(t) = \frac{200}{C'''e^{-t/50}+1}$

$P(0) = 100$

$\hookrightarrow 100 = \frac{200}{C'''+1}$

$100C''' + 100 = 200$

$100C''' = 100$

$C''' = 1$

$P(t) = \frac{200}{e^{-t/50}+1}$

$P(60) = \frac{200}{e^{-t/50}+1} = \frac{200}{e^{-60/50}+1} \approx 153.7$

In 2000 there will be approximately 153.7 million people.

(28) $\frac{dx}{dt} = 0.0001x^2 - .01x$

a) $(0, 25) \rightarrow x(0) = 25$

$\cancel{\frac{dx}{dt} = 0.0001x^2 - 0.01x = (.01x)^2 - .01x = .01x(.01x-1)}$

$\cancel{\int \frac{dx}{.01x(.01x-1)} = \int dt}$

$$\frac{1}{.01x(.01x-1)} = \frac{A}{.01x} + \frac{B}{(.01x-1)} = \frac{.01xA - A + .01xB}{.01x(.01x-1)}$$

so $.01x(A+B) + (-1)A = 1$

$A+B=0$, $-A=1$

$-1+B=0$, $A=-1$

$B=1$

$$\frac{1}{.01x(.01x-1)} = \frac{-1}{.01x} + \frac{1}{(.01x-1)}$$

$\cancel{-\frac{1}{.01}\int \frac{1}{x}dx + \int \frac{1}{.01x-1} = \int dt}$

$\cancel{-\frac{1}{.01}\ln|x| + \frac{1}{.01}\ln|.01x-1| = t+C}$

$\frac{1}{.01}\left(\ln\left|\frac{.01x-1}{x}\right|\right) = t+C$

$e^{\left(\ln\left|\frac{.01x-1}{x}\right|\right)} = e^{(.01t+C')}$

$\left|\frac{.01x-1}{x}\right| = e^{1.01t}$

$.01x - 1 = xC''e^{.01t}$

$x(-C''e^{.01t} + .01) = 1$

$x = C''e^{-.01t} + \frac{1}{.01} \rightarrow 100$

$x(0) = 25$

$25 = C''e^{-.01(0)} + 100$

$25 = C'' + 100$

$-75 = C''$

$\cancel{x(t) = -75e^{-.01t}} + 100$

$\lim (x(t)) = 0 \rightarrow$ The alligators will eventually die out.

b) $x(0) = 150 \rightarrow 150 = C''e^{-.01(0)} + 100$

$150 = C'' + 100$

$50 = C''$

$x(t) = 50e^{-.01t} + 100 \rightarrow 0 = 50e^{-.01t} + 100$

$-100 = 50e^{-.01t}$

$\ln(-1/2) = -.01t$

$t \cong 110$

They will die out in about 110 months