Unit ball: surface area and volume

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1 Definitions

$V(S)$:= the volume of set S,
B_n := the unit ball in $\mathbb{R}^n$,
$V_n := V(B_n)$:= the volume of the unit sphere in $\mathbb{R}^n$,
ω_n := the area of the unit sphere in $\mathbb{R}^n$,
$\hat{\mathbf{n}}$:= the outward unit normal,
$\mathbf{x}$:= identity function (on $\mathbb{R}^n$),
$\int := \int_{B_n}$, and
$\oint := \int_{\partial B_n}$.

2 Surface Area

The surface area of a sphere is related to its volume by Gauss's law:

$$\omega_n = \oint \hat{\mathbf{n}} \cdot \mathbf{x} = \int \nabla \cdot \mathbf{x} = \int n = nV_n.$$

A more elementary way to see this is:

$$V_n = \int_{r=0}^{1} \omega_n r^{n-1}\, dr = \frac{\omega_n}{n} r^n \Big|_{r=0}^{1} = \frac{\omega_n}{n}.$$

3 Volume

The volume of a radius-r ball is

$$V(rB_n) = r^n V_n.$$

A recurrence relation for the volume is given by:

$$V_{n+1} = V_n \underbrace{\int_{-1}^{1} (\sqrt{1-t^2})^n dt}_{\text{Call } I_{n+1}}.$$

. Let $t = \sin t$, $-\pi/2 \le t \le \pi/2$.

$$\begin{aligned} I_{n+1} &= \int_{-\pi/2}^{\pi/2} \cos^{n+1} \\ &= \int_{-\pi/2}^{\pi/2} \cos^n \cos \\ &= n \int_{-\pi/2}^{\pi/2} \cos^{n-1} \sin^2 \\ &= n\left(\int_{-\pi/2}^{\pi/2} \cos^{n-1} - \int_{-\pi/2}^{\pi/2} \cos^{n+1} \right); \end{aligned}$$

Solving for I_{n+1} yields the recurrence relation

$$I_{n+1} = \frac{n}{n+1} I_{n-1},$$

i.e.,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

n	I_n	V_n	ω_n
0	π	1	0
1	2	2	2
2	$\frac{\pi}{2}$	π	2π
3	$\frac{4}{3}$	$\frac{4}{3}\pi$	4π
$2m$	$\pi\frac{(2m-1)!!}{(2m)!!} = \pi\frac{(2m)!}{(2^m m!)^2}$	$\frac{(2\pi)^m}{(2m)!!} = \frac{\pi^m}{m!}$	$\frac{(2\pi)^m}{(2(m-1))!!}$
$2m+1$	$2\frac{(2m)!!}{(2m+1)!!} = 2\frac{(2^m m!)^2}{(2m)!(2m+1)}$	$2\frac{(2\pi)^m}{(2m+1)!!}$	$2\frac{(2\pi)^m}{(2m-1)!!}$
n		$\frac{\pi^{n/2}}{(n/2)!}$	$2\frac{\pi^{n/2}}{\Gamma(n/2)}$

Here $n! = \Gamma(n+1)$. The double factorial !! is defined for even integers to be the product of all evens up to the argument and for odd integers to be the product of all odds up to the argument. So $(2m)!! = 2^m m!$ and $(2m-1)!! = \frac{(2m)!}{(2m)!!}$. For half-integer values of n, using

$$(m-1/2)! = \Gamma(m+1/2) = \sqrt(\pi)\frac{(2m-1)!!}{2^m},$$

the same formula works for both even and odd numbers of dimensions.

References

[1] http://en.wikipedia.org/wiki/Factorial#Double_factorial

[2] http://en.wikipedia.org/wiki/Unit_sphere