Consistency of Discontinuous Galerkin methods

by Alec Johnson, November 3, 2010

1 Taylor series.

We recall Taylor's theorem. A simple version in one dimension says that for $F(t)$ smooth on $[0,1]$

$$F(1) = \sum_{s=0}^{r-1} \frac{1}{s!} D^s F(0) + \frac{1}{r!} D^r F(t),$$

where $t \in [0,1]$. To obtain the general Taylor's theorem let $F(t) = f(\mathbf{x}_0 + t\Delta\mathbf{x})$. Then D becomes $(\Delta\mathbf{x}\cdot\nabla)$ and the 1-D Taylor's theorem now reads as a general multidimensional Taylor's theorem:

$$f(\mathbf{x}_0 + \Delta\mathbf{x}) = \sum_{s=0}^{r-1} \frac{1}{s!} (\Delta\mathbf{x}\cdot\nabla)^s f(\mathbf{x}_0) + \frac{1}{r!} (\Delta\mathbf{x}\cdot\nabla)^r f(\tilde{\mathbf{x}}),$$

where $\tilde{\mathbf{x}} \in [\mathbf{x}_0, \mathbf{x}_0 + \Delta\mathbf{x}]$. Another form comes from the multinomial expansion $(\sum_{i=1}^n y_i)^s = \sum_{|\mathbf{k}|=s} \frac{s!}{\mathbf{k}!} \mathbf{y}^{\mathbf{k}}$, where $|\mathbf{k}| := |\mathbf{k}|_1 = \sum_{i=1}^n k_i$, $\mathbf{k}! := \prod_{i=1}^n k_i!$, and $\mathbf{y}^{\mathbf{k}} := \prod_{i=1}^n y_i^{k_i}$ ($\mathbf{k}$ is a vector of nonnegative integers). So the multinomial form of Taylor's theorem reads

$$\begin{aligned} f(\mathbf{x}_0+\Delta\mathbf{x}) = &\sum_{|\mathbf{k}|<r} \frac{1}{\mathbf{k}!} \Delta\mathbf{x}^{\mathbf{k}} \nabla^{\mathbf{k}} f(\mathbf{x}_0) \\ &+ \sum_{|\mathbf{k}|=r} \frac{1}{\mathbf{k}!} \Delta\mathbf{x}^{\mathbf{k}} \nabla^{\mathbf{k}} f(\tilde{\mathbf{x}}) \end{aligned}$$

where $\nabla^{\mathbf{k}} := \partial_1^{k_1} \cdots \partial_n^{k_n}$. In particular, taking $\mathbf{x}_0 = 0$ and $\Delta\mathbf{x} = \mathbf{x}$, there exist constants $c_{\mathbf{k}}$ ($= \frac{1}{\mathbf{k}!} \nabla^{\mathbf{k}} f(0)$) and functions $\tilde{c}_{\mathbf{k}}(\mathbf{x})$ bounded by $\left\| \frac{1}{\mathbf{k}!} \nabla^{\mathbf{k}} f \right\|_\infty$ such that

$$f(\mathbf{x}) = \sum_{|\mathbf{k}|<r} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} + \sum_{|\mathbf{k}|=r} \tilde{c}_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}.$$

2 Consistent Projection.

Discontinuous Galerkin (DG) works by in each mesh cell projecting onto the space V^N of polynomials of degree at most N using integrals approximated by Gaussian quadrature. We wish to show that for smooth solutions the error on a mesh cell of width $\Delta\mathbf{x}$ is of order $\mathcal{O}(\Delta\mathbf{x}^{N+1})$. Toward this end we show that the projection of a smooth function onto a mesh cell using Gaussian quadrature has error of order $\mathcal{O}(\Delta\mathbf{x}^{N+1})$.

Remark. When the statement "this method is Nth order accurate" is made, the novice naturally wonders whether this means that the error term is of order N or $N+1$. We compromise. The error over one time step (in each cell or globally) is of order $N+1$ and the error over a fixed time interval (at a given point or globally) is of order N.

Projection.

Definition 2.1. We say that $u_P \in V^N$ is the **projection** of u onto V^N relative to the (possibly degenerate, i.e. not strictly positive definite) inner product $\langle \cdot, \cdot \rangle$ if for any test function ϕ in V^N, $\langle u, \phi \rangle = \langle u_P, \phi \rangle$. In particular, if ϕ_k is a basis for V^N with reciprocal basis ϕ^k then the projection of u onto V^N is $u = u^k \phi_k$ where $u^k = \langle u, \phi_k \rangle$.

L2 projection. Fix the spatial domain to be a mesh cell C. Let V^N denote the space of polynomials of degree at most N (on C). The L2 projection of $u(\mathbf{x})$ is defined as the element $u_P \in V^N$ which minimizes $\|u - u_P\|_2$, where the 2-norm is defined by the inner product $\langle u, v \rangle := \fint_C uv$, where $\fint_C := (1/\mathrm{vol}(C)) \int_C$ is the averaging integeral. I claim that u_P is the orthogonal projection of u onto V^N, i.e. the vector u_P in V^N which for all v in V^N satisfies $\langle u - u_P, v \rangle = 0$, i.e. $\langle u - u_P, v - u_P \rangle = 0$. Indeed, then the pythagorean theorem gives $\|u - v\|^2 = \|u - u_P\|^2 + \|v - u_P\|^2 \ \ \forall v \in V^N$, which also shows that u_P is unique.

General Projection. More generally, orthogonal projection of a vector onto a subspace in a given inner product (that is strictly positive definite when restricted to the subspace) finds the element of the subspace that minimizes the seminorm (induced by the inner product) of the difference between the vector and its projection.

Orthogonal projection onto a basis. Let $\phi_k(\mathbf{x})$ be polynomial basis functions which span V^N. Then we can write $u_P(\mathbf{x}) = \sum_{k=1}^{N_p} u^k \phi_k(\mathbf{x})$. Let $\phi^k(\mathbf{x})$ denote the reciprocal basis of V^N defined by the requirement that $\langle \phi_i, \phi^j \rangle = \delta_i^j$. Then the coefficients of the expansion are $u^k = \langle u, \phi^k \rangle$ (which shows the existence of the orthogonal projection onto a finite-dimensional subspace).

Gaussian quadrature. Gaussian quadrature approximates $\fint_C f$ with a weighted sum of sampled values: $\fint_C^{\mathrm{GQ}} f := \sum_{i=1}^m w_i f(\mathbf{x}_i^*)$. For a one-dimensional interval, Gaussian quadrature with m optimally chosen points and weights exactly integrates polynomials of degree at most $M := 2m - 1$. Suppose that f is a polynomial of degree at most M and C is a rectangular domain in n-dimensional space. Then we

can integrate over C using repeated 1-dimensional integrals. Each iterated integral (exactly) integrates a polynomial of degree at most M, so for each iterated integral we can use 1-dimensional Gaussian quadrature $\int^{\mathrm{GQ}}_{[-1,1]} f := \sum_{i=1}^m w_i f(\xi_i^*)$ accurate for polynomials of degree at most M. In particular, letting the canonical rectangular mesh cell be $\tilde{C} := [-1,1]^n$ (the nth cartesian power of the canonical interval $[-1,1]$),

$$\begin{aligned}
&\fint_{\mathbf{x}\in C} f(\mathbf{x}) = \fint_{\boldsymbol{\xi}\in\tilde{C}} \tilde{f}(\boldsymbol{\xi}) \\
&= \fint_{\xi_1\in[-1,1]} \cdots \fint_{\xi_n\in[-1,1]} \tilde{f}(\boldsymbol{\xi}) \\
&= \int^{\mathrm{GQ}}_{\xi_1\in[-1,1]} \cdots \int^{\mathrm{GQ}}_{\xi_n\in[-1,1]} \tilde{f}(\boldsymbol{\xi}) \\
&= \sum_{i_1=1}^m \cdots \sum_{i_n=1}^m w_{i_1}\cdots w_{i_n} \tilde{f}(\xi^*_{i_1},\ldots,\xi^*_{i_n}),
\end{aligned}$$

showing that a quadrature rule that exactly averages multivariate polynomials of degree at most M over the canonical mesh cell $\tilde{C}$ is the "nth tensor power" of a one-dimensional Gaussian quadrature (tensor power of weights, cartesian power of points) that exactly averages polynomials of degree at most M over the canonical interval $[-1,1]$.

Tensor product basis. Suppose that the polynomials ϕ^k are a basis for the space of polynomials of degree at most N (restricted to the canonical interval $[-1,1]$). Then we can define the tensor product bases $\phi^{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ on the canonical cell $\tilde{C}$ by $\phi^{\mathbf{k}}(\boldsymbol{\xi}) := \phi^{k_1}(\xi_1)\cdots\phi^{k_n}(\xi_n)$ and $\phi_{\mathbf{k}}(\boldsymbol{\xi}) := \phi_{k_1}(\xi_1)\cdots\phi_{k_n}(\xi_n)$. Observe that

$$(\phi_{\mathbf{k}'}, \phi^{\mathbf{k}}) = \delta^{\mathbf{k}}_{\mathbf{k}'} := \delta^{k_1}_{k_1'}\cdots\delta^{k_n}_{k_n'}.$$

Legendre (orthogonal) polynomials. A standard basis for V^N is the monic monomials. On the canonical interval $[-1,1]$ it is $\{\xi^k\}_{k=0}^N = \{0, \xi, \xi^2, \ldots, \xi^N\}$. On the canonical mesh cell $\tilde{C}$ the basis of monic monomials is the tensor product of the 1-dimensional monic monomials, $\{\boldsymbol{\xi}^{\mathbf{k}}\}^N_{|\mathbf{k}|=0} = \{\xi_1^{k_1}\cdots\xi_n^{k_n}\}^N_{|\mathbf{k}|=0}$. We can use the Gram-Schmidt procedure to define an orthonormal basis ψ^k satisfying $(\psi_k, \xi^j) = 0$ for $j < k$. The Legendre polynomials are such a basis, and any other such basis consists of rescaled versions of the Legendre polynomials. The tensor product basis for Legendre polynomials is an orthonormal basis and satisfies $(\psi_{\mathbf{k}}, \boldsymbol{\xi}^{\mathbf{j}}) = 0$ if $\mathbf{j} \not\geq \mathbf{k}$, i.e. if $j_i < k_i$ for some i.

Taylor expansion. Taylor expand $u(\mathbf{x})$ around a point $\mathbf{x}_0$ in the mesh cell. Without loss of generality $\mathbf{x}_0 = \mathbf{0}$. Then the Taylor expansion of order N says

$$u(\mathbf{x}) = u_T(\mathbf{x}) + u_R(\mathbf{x})$$

where the Taylor series approximation and remainder (error) terms are

$$u_T(\mathbf{x}) = \sum_{|\mathbf{k}|\leq N} c_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}, \qquad u_R(\mathbf{x}) = \sum_{|\mathbf{k}|=N+1} \tilde{c}_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}.$$

Projection. To project u onto V^N transform from physical coordinates $\mathbf{x}$ in cell C to canonical coordinates $\boldsymbol{\xi}$ in canonical cell $\tilde{C}$, where $\mathbf{x} = \boldsymbol{\xi} * \Delta\mathbf{x}$, i.e. $x_i = \xi_i * \Delta x_i$. Making this substitution in the Taylor expansion,

$$\begin{aligned}
u_T(\boldsymbol{\xi}) &= \sum_{|\mathbf{k}|\leq N} c_{\mathbf{k}}\Delta\mathbf{x}^{\mathbf{k}}\boldsymbol{\xi}^{\mathbf{k}}, \\
u_R(\boldsymbol{\xi}) &= \sum_{|\mathbf{k}|=N+1} \tilde{c}_{\mathbf{k}}\Delta\mathbf{x}^{\mathbf{k}}\boldsymbol{\xi}^{\mathbf{k}}.
\end{aligned}$$

So the coefficients of the projection are $u^{\mathbf{k}} := u_T^{\mathbf{k}} + u_R^{\mathbf{k}}$ where

$$\begin{aligned}
u_T^{\mathbf{j}} &= \langle u_T(\mathbf{x}), \phi^{\mathbf{j}}(\mathbf{x})\rangle = \sum_{|\mathbf{k}|\leq N} c_{\mathbf{k}}\langle \mathbf{x}^{\mathbf{k}}, \phi^{\mathbf{j}}(\mathbf{x})\rangle \\
&= \sum_{|\mathbf{k}|\leq N} c_{\mathbf{k}}\Delta\mathbf{x}^{\mathbf{k}}\langle \boldsymbol{\xi}^{\mathbf{k}}, \phi^{\mathbf{j}}(\boldsymbol{\xi})\rangle \quad \text{and similarly} \\
u_R^{\mathbf{j}} &= \langle u_R(\mathbf{x}), \phi^{\mathbf{j}}(\mathbf{x})\rangle = \sum_{|\mathbf{k}|=N+1} \tilde{c}_{\mathbf{k}}\langle \mathbf{x}^{\mathbf{k}}, \phi^{\mathbf{j}}(\mathbf{x})\rangle \\
&= \sum_{|\mathbf{k}|=N+1} \tilde{c}_{\mathbf{k}}\Delta\mathbf{x}^{\mathbf{k}}\langle \boldsymbol{\xi}^{\mathbf{k}}, \phi^{\mathbf{j}}(\boldsymbol{\xi})\rangle.
\end{aligned}$$

Remark 2.2. If $\phi^{\mathbf{j}} = \psi^{\mathbf{j}}$ (the Legendre polynomials) then the fact that $\langle \boldsymbol{\xi}^{\mathbf{k}}, \psi^{\mathbf{j}}\rangle = 0$ if $\mathbf{k} \not\geq \mathbf{j}$ means that $u^{\mathbf{j}} = \mathcal{O}(\Delta\mathbf{x}^{\mathbf{j}})$, i.e., as you refine the mesh higher-order Legendre coefficients of the projection decay with the order of the degree of their polynomial.

Proposition 2.1. *Let $u(\mathbf{x})$ be a smooth function. The error of the projection onto V^N of a smooth function $u(\mathbf{x})$ is of the same order $\mathcal{O}(\Delta\mathbf{x}^{N+1})$ as the order of the remainder in the Taylor expansion of $u(\mathbf{x})$.*

Proof. Let $u_P = \langle u, \phi^{\mathbf{k}}\rangle\phi_{\mathbf{k}}$ be the projection of u onto V^N. The error is $(u_P - u) = (u_P - u_T) + (u_T - u)$. But since u_T is its own projection and projection is linear, this is just the projection of the remainder minus the remainder, both of which are order of the remainder. That is, $(u_T - u) = -u_R = \mathcal{O}(\Delta\mathbf{x}^{N+1})$ and $(u_P - u_T) = \langle u - u_T, \phi^{\mathbf{k}}\rangle\phi_{\mathbf{k}} = \langle u_R, \phi^{\mathbf{k}}\rangle\phi_{\mathbf{k}} = \mathcal{O}(\Delta\mathbf{x}^{N+1})$ since $\phi_{\mathbf{k}} = \mathcal{O}(1)$. □

The foregoing analysis holds regardless of whether the inner product used to define the projection is integration, a Gaussian quadrature rule, or any inner product that is positive definite on V^N.

3 Proof of consistency

Definition 3.1. We say that an evolution equation and a numerical method are **consistent** of order $\mathcal{O}(\Delta\mathbf{x}^{N+1})$ if the residual (i.e. the error of the rate of change of the solution) when the exact solution to the evolution equation is substituted into the numerical method is of order $\mathcal{O}(\Delta\mathbf{x}^{N+1})$.

Proposition 3.1. *Suppose that $u(\mathbf{x},t)$ satisfies the weak form of the evolution equation*

$$d_t \int (u\phi) + \oint (\mathbf{n}\cdot f\phi) = \int f\cdot\nabla\phi + \int s\phi$$

for all test functions ϕ. We wish to show that u approximately satisfies the numerical scheme

$$d_t \int_C^{\mathrm{GQ}} (u\phi^{\mathbf{k}}) + \oint_{\partial C}^{\mathrm{GQ}} (\mathbf{n}\cdot f\phi^{\mathbf{k}}) = \int_C^{\mathrm{GQ}} f\cdot\nabla\phi^{\mathbf{k}} + \int_C^{\mathrm{GQ}} s\phi^{\mathbf{k}}.$$

where $\phi^{\mathbf{k}}$ is a basis for V^N.

We have not yet made any use of the fact that Gaussian quadrature agrees with integration on V^N; we have only used that the inner product it defines is positive definite on V^N. To prove consistency we will need to use some sort of relationship between the integrals in the weak evolution equation and the numerical quadratures in the numerical scheme; in particular, we will use that Guassian quadrature agrees with integration on V^N.

The essence of the proof is that Gaussian quadrature is exact on the Taylor approximation and that the projection of the Taylor approximation is itself.

Justification of consistency. We can consider each term individually. Consider the boundary integral term. Recall that $\phi^{\mathbf{k}}(\mathbf{x}) = \mathcal{O}(1)$ (because we define $\phi^{\mathbf{k}}(\mathbf{x})$ by its values $\phi^{\mathbf{k}}(\boldsymbol{\xi})$ in the canonical mesh cell). Let $u(\mathbf{x},t)$ be a smooth solution satisfying the evolution equation. So $f(u)$ and $s(u)$ are also smooth. So we can Taylor expand them in each mesh cell, e.g. $f = f_T + \mathcal{O}(\Delta\mathbf{x}^{N+1})$ where $f_T \in V^N$ is a Taylor expansion. we have e.g. $\oint_C(\mathbf{n}\cdot f\phi^{\mathbf{k}}) = \oint_C(\mathbf{n}\cdot f_T\phi^{\mathbf{k}}) + \mathcal{O}(\Delta\mathbf{x}^{N+1})$ and likewise $\oint_{\partial C}^{\mathrm{GQ}}(\mathbf{n}\cdot f\phi^{\mathbf{k}}) = \oint_{\partial C}^{\mathrm{GQ}}(\mathbf{n}\cdot f_T\phi^{\mathbf{k}}) + \mathcal{O}(\Delta\mathbf{x}^{N+1})$. But $\oint_{\partial C}^{\mathrm{GQ}}(\mathbf{n}\cdot f_T\phi^{\mathbf{k}}) = \oint_{\partial C}(\mathbf{n}\cdot f_T\phi^{\mathbf{k}})$ *if we require that the Guassian quadrature used to calculate the surface integrals exactly integrates polynomials of degree at most $2N$.* So

$$\oint_{\partial C}^{\mathrm{GQ}} (\mathbf{n}\cdot f\phi^{\mathbf{k}}) = \oint_C (\mathbf{n}\cdot f\phi^{\mathbf{k}}) + \mathcal{O}(\Delta\mathbf{x}^{N+1}).$$

The proofs for the other terms are similar. Note that for the term $\int_C^{\mathrm{GQ}} f\cdot\nabla\phi^{\mathbf{k}}$ it is sufficient to use a quadrature rule which exactly integrates polynomials of degree $2N-1$ (because $\nabla\phi^{\mathbf{k}}$ is of order $N-1$). So we conclude that

$$d_t \int_C^{\mathrm{GQ}} (u\phi^{\mathbf{k}}) = d_t \int (u\phi^{\mathbf{k}}) + \mathcal{O}(\Delta\mathbf{x}^{N+1})$$

(where $\phi^{\mathbf{k}}$ may actually be any $\mathcal{O}(1)$ test function), which is basically what we mean when we say that the rate of change of the solution in V^N has error of order $\mathcal{O}(\Delta\mathbf{x}^{N+1})$. □

Remark 3.2. Note that the fact that in the Legendre basis $\int_C^{\mathrm{GQ}}(u\phi^{\mathbf{k}}) = \mathcal{O}(\Delta\mathbf{x}^{\mathbf{k}})$ is completely irrelevent to this proof (and in most other (e.g. nodal) bases there are no such decaying higher modes).

Remark 3.3. Note that we gave no consideration to the numerical flux function at the boundaries. For consistency and stability to imply convergence the numerical flux function needs to depend continuously on the states on each side of the boundary. For consistency we also require that the numerical flux function agree with the physical flux function when the states on both sides of the boundary are equal, e.g. for smooth data as obtains in the proof of consistency.