A Derivation of the One-fluid Plasma Model from the Multi-fluid Model.

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1 Definition of quantities.

s = species index; typically $s \in \{e, i\}$, where e denotes electron and i denotes ion.

All quantities with a subscript s are specific to the species s.

q_s = charge of a particle
m_s = mass of a particle
n_s = particle number density
$\mathbf{u}_s$ = average particle velocity
$\sigma_s := n_s q_s$ = charge density (Some call this ρ_{qs}.)
$\rho_s := n_s m_s$ = mass density (Some call this ρ_{ms}.)
$\mathbf{J}_s := \sigma_s \mathbf{u}_s$ = charge flux = current density (Some call this $\mathbf{J}_{qs}$.)
$(\rho_s \mathbf{u}_s)$ = mass flux = momentum density (Some call this $\mathbf{J}_{ms}$.)
$\mathbb{P}_s$ = pressure tensor
p_s = scalar pressure
ε_s = gas-dynamic energy per volume
T_s = temperature
S_s = entropy
$\mathbf{R}_s$ = drag force (per volume) on species s due to collisions with other species.
$\mathbf{q}$ = heat flux vector
Q_s = rate of heat transfer from other species to species s (due to collisions).
K_s = rate of energy transfer from other species to species s due to collisions $= \mathbf{R}_s \cdot \mathbf{u}_s + Q_s$.

Quantities without a subscript apply to the plasma as a whole.

$\sigma := \sum_s \sigma_s$ = net charge density
$\rho := \sum \rho_s$ = net mass density
$\mathbf{J} := \sum \mathbf{J}_s$ = net current density
$(\rho\mathbf{u}) := \sum \rho_s \mathbf{u}_s$ = net momentum density (i.e. $\mathbf{u} := \frac{\sum \rho_s \mathbf{u}_s}{\rho}$.)
$\mathbb{P} := \sum_s \mathbb{P}_s$ = pressure tensor

$\mathbf{E}$ = electric field
$\mathbf{B}$ = magnetic field

V_s = some volume element convected by $\mathbf{u}_s$.
Let $\int := \int_{V_s}$ and let $\oint := \int_{\partial V_s}$.
Let $d_t^s := \partial_t + \mathbf{u}_s \cdot \nabla$ denote the *convective derivative.*
Let $\delta_t^s := \alpha \mapsto (\partial_t \alpha + \nabla \cdot (\mathbf{u}_s \alpha))$ denote the *conservative derivative.*

2 Conservation of mass.

We assume that particles are conserved. This means that the number of particles of species s in a volume convected by $\mathbf{u}_s$ remains constant:
$d_t \int_{V_s} n_s = 0 \iff \delta_t^s(n_s) = 0$
$\iff \boxed{\partial_t(n_s) + \nabla \cdot (\mathbf{u}_s n_s) = 0}$
Multiplying by q_s gives charge conservation:
$\partial_t(\sigma_s) + \nabla \cdot (\mathbf{J}_s) = 0$
Multiplying by m_s gives mass conservation:
$\partial_t(\rho_s) + \nabla \cdot (\rho_s \mathbf{u}_s) = 0$

Deriving one-fluid equation.

If we sum over all species s, we get equations for conservation of net charge and total mass:
$\partial_t \sigma + \nabla \cdot \mathbf{J} = 0$
$\boxed{\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0}$

3 Conservation of momentum.

To write conservation of momentum, we must identify the sources of forces. The fundamental forces are electromagnetic force, nuclear force, and gravitational force. We are ignoring gravity.

We model the net force on a species as the sum of a macroscopic electromagnetic field (averaged over a region roughly the size of a Debye sphere) plus a pressure force (due to collisions with particles of the same species, which cancel everywhere in a convected test volume except at the boundary) plus a resistive drag force due to collisions with other species:

$d_t \int m_s n_s \mathbf{u}_s = -\oint \hat{n} \cdot \mathbb{P}_s + \int (q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B})) + \int \mathbf{R}_s$
$\boxed{\partial_t(\rho_s \mathbf{u}_s) + \nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s) = -\nabla \cdot \mathbb{P}_s + \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \mathbf{R}_s}$

Deriving one-fluid momentum equation.

We sum over all species. Since total momentum is conserved for any collision process, $\sum_s \mathbf{R}_s = 0$. For the linear terms the sum effectively replaces each species quantity with the corresponding net quantity.

We handle the nonlinear term in the standard manner of statistical mechanics, by absorbing the nonlinearity into the pressure tensor. We define:
$\boxed{\mathbf{w}_s := \mathbf{u}_s - \mathbf{u}}$
$\mathbf{w}_s$ is the average velocity of species s relative to the net velocity $\mathbf{u}$, also known as the *diffusive velocity* for species s.
Then $\sum_s \rho_s \mathbf{w}_s = \sum_s \rho_s (\mathbf{u}_s - \mathbf{u}) = \sum_s \rho_s \mathbf{u}_s - \sum_s \rho_s \mathbf{u} = \rho\mathbf{u} - \rho\mathbf{u} = 0.$
So $\sum_s \rho_s \mathbf{u}_s \mathbf{u}_s = \sum_s \rho_s (\mathbf{u} + \mathbf{w}_s)(\mathbf{u} + \mathbf{w}_s)$
$= \rho\mathbf{u}\mathbf{u} + \cancel{\sum_s \rho_s \mathbf{u}\mathbf{w}_s} + \cancel{\sum_s \rho_s \mathbf{w}_s \mathbf{u}} + \sum_s \rho_s \mathbf{w}_s \mathbf{w}_s$

Define $\mathbb{P}_{0s} := \mathbb{P}_s + \rho_s \mathbf{w}_s \mathbf{w}_s$
Define $\boxed{\mathbb{P}_0 := \sum_s \mathbb{P}_{0s} = \mathbb{P} + \sum_s \rho_s \mathbf{w}_s \mathbf{w}_s}$

So the net (i.e. one-fluid) momentum equation is:
$\boxed{\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla \cdot \mathbb{P}_0 + \sigma \mathbf{E} + \mathbf{J} \times \mathbf{B}}$

4 Conservation of energy.

The total energy of the system is the energy of the electromagnetic field plus the (kinetic) energy in each species. (Again we are ignoring gravity.) The kinetic energy ε_s of species s is the sum of the macroscopic kinetic energy $\frac{1}{2}\rho_s u_s^2$ plus the thermal energy per unit volume, $\rho_s e_s$, where e_s is the thermal energy per unit mass:
$\boxed{\varepsilon_s = \rho_s e_s + \frac{1}{2}\rho_s u_s^2}$

The gas-dynamic energy of the species s within the convected test volume V_s is changed as a result of work performed by electromagnetic and pressure forces, heat flow within the species, and interaction (collisions) with other species. Energy balance for species s is thus:
$d_t \int \varepsilon_s = -\oint(\hat{n} \cdot \mathbb{P}_s \cdot \mathbf{u}_s) + -\oint \hat{n} \cdot (\mathbf{q}_s)$
$\quad + \int \mathbf{u}_s \cdot (n_s q_s(\mathbf{E} + \cancel{\mathbf{u}_s \times \mathbf{B}})) + \int K_s.$
Recall that K_s denotes the rate of transfer of energy from other species to s due to collisions.
Cast as a differential equation, this says:
$\partial_t \varepsilon_s + \nabla \cdot (\mathbf{u}_s \varepsilon_s) = -\nabla \cdot (\mathbb{P}_s \cdot \mathbf{u}_s) + -\nabla \cdot \mathbf{q}_s + \mathbf{J}_s \cdot \mathbf{E} + K_s$

4.1 Deriving one-fluid energy equation.

We again sum over all species, as we did for the one-fluid momentum balance. Since total kinetic energy is conserved for any collisional process, $\sum_s K_s = 0$. We again deal with nonlinear terms using $\mathbf{u}_s = \mathbf{u} + \mathbf{w}_s$. In deriving the one-fluid momentum equation we linearized by absorbing the nonlinearity from the inertial term into the definition of pressure. Here we linearize by absorbing anything we don't want into the definition of the heat flux.
The equation we want is:
$\partial_t \varepsilon_s + \nabla \cdot (\mathbf{u}\varepsilon_s) = -\nabla \cdot (\mathbb{P}_{0s} \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}_{0s} + \mathbf{J}_s \cdot \mathbf{E} + K_s$
(Summing over all species gives this same equation without the s subscripts, which is the desired 1-fluid equation.)
Comparing this with the actual conservation of energy equation for species s tells how we need to define $\mathbf{q}_{0s}$.
Taking the difference of the two equations and summing over species gives:
$\nabla \cdot \sum_s (\mathbf{w}_s \varepsilon_s + \mathbb{P}_s \cdot \mathbf{u}_s - \mathbb{P}_{0s} \cdot \mathbf{u} + \mathbf{q}_s - \mathbf{q}_{0s}) = 0$
(1-fluid heat flux requirement)

There is a great deal of freedom here in choosing how to work backwards to a definition of $\mathbf{q}_{0s}$. Of course we could just throw out the divergence and the sum in the previous equation and satisfy the book-keeping requirement of the exact 1-fluid equation, but we seek a definition of $\mathbf{q}_{0s}$ that minimizes the size of the terms appearing in $\mathbf{q}_{0s} - \mathbf{q}_s$. The most obvious requirement to satisfy is to express the heat flux requirement purely in terms of relative velocities, eliminating all references to absolute velocity. You can throw out the divergence at this point, since I don't see any way to use the divergence to make anything cancel.
$\sum_s \left(\mathbf{w}_s \varepsilon_s + (\mathbb{P}_s - \mathbb{P}_{0s}) \cdot \mathbf{u} + \mathbb{P}_s \cdot \mathbf{w}_s + \mathbf{q}_s - \mathbf{q}_{0s}\right) = 0$
Using $\sum_s \rho_s \mathbf{w}_s = 0$ and $\mathbb{P}_{0s} - \mathbb{P}_s = \rho_s \mathbf{w}_s \mathbf{w}_s$, get:
$\sum_s \left(\mathbf{w}_s \rho_s (e_s + \cancel{\tfrac{1}{2}u^2} + \cancel{\mathbf{w}_s \cdot \mathbf{u}} + \tfrac{1}{2} w_s^2)\right.$
$\quad \left. - \cancel{\rho_s \mathbf{w}_s \mathbf{w}_s \cdot \mathbf{u}} + \mathbb{P}_s \cdot \mathbf{w}_s + \mathbf{q}_s - \mathbf{q}_{0s}\right) = 0$
So we can make the definition
$\boxed{\mathbf{q}_{0s} := \mathbf{q}_s + \mathbf{w}_s \rho_s (e_s + \tfrac{1}{2} w_s^2) + \mathbb{P}_s \cdot \mathbf{w}_s}$
This says that the contribution of each species to the effective heat flux is its own heat flux plus its relative energy flux plus the relative work performed by its internal pressure. (The reason no interactions with other species are incorporated here is that in the average they cancel.)

So the net (i.e. one-fluid) energy equation is:
$\boxed{\partial_t \varepsilon + \nabla \cdot (\mathbf{u}\varepsilon) = -\nabla \cdot (\mathbb{P}_0 \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}_0 + \mathbf{J} \cdot \mathbf{E}}$
Recall here that ε, $\mathbb{P}_0$, $\mathbf{q}_0$, and $\mathbf{J}$ are sums of the corresponding quantities for each species.

Hopefully $\mathbf{q}_0 - \mathbf{q} := \sum_s \mathbf{w}_s \rho_s (e_s + \frac{1}{2} w_s^2) + \mathbb{P}_s \cdot \mathbf{w}_s$ and $\mathbb{P}_0 - \mathbb{P} = \sum_s \rho_s \mathbf{w}_s \mathbf{w}_s$ are small and well-behaved.