A Kinetic Derivation of Plasma Laws

by Alec Johnson, December 2006

1 Kinetic Quantities

A plasma is made up of a large number of charged particles. Each particle has a position $\mathbf{r}$ and velocity $\mathbf{v} := \dot{\mathbf{r}}$ that depend on time t. Each particle also has a mass m and a charge q. Assume that for a given species of particle (say p) particle positions and velocities are distributed according to the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. That is, the number of particles in an infinitesimal phase space volume element $d^3\mathbf{r}d^3\mathbf{v}$ is $fd^3\mathbf{r}d^3\mathbf{v}$.

When we need to specify the species of particle to which a quantity refers, we will use a species index. When a species-dependent variable lacks a species index, it is assumed that the default species of interest p is being referred to. (e.g. $f = f_p$ by default.) So f_s will typically refer to the distribution function of a second species.

The position of a particle in phase space is $R := (\mathbf{r}, \mathbf{v})$. So the velocity in phase space is $V := \dot{R} = (\dot{\mathbf{r}}, \dot{\mathbf{v}}) =: (\mathbf{v}, \mathbf{a})$. $\mathbf{a}$ is the force per mass. So $\mathbf{a}$ is a function of t, $\mathbf{r}$, and $\mathbf{v}$. We will assume that this function is smooth.

It is useful to divide the forces on particles into two types: collision forces and macroscopic forces. The advantage of this division is that it allows us to say that $\mathbf{a}$ is given by the macroscopic forces (which are smoothly varying and are independent of individual particles). The cost of this division is that collisions must be modeled separately. Collisions cause the velocity of particles to change abruptly.

2 Liouville equation

Let's begin by assuming that we can ignore particle collisions. Since the velocity in phase space is a smooth function of position in phase space, we can regard particles as flowing through phase space with velocity V. This means that we can think of the probability density as a fluid flowing through phase space. Since particles are conserved, the probability density of particles is conserved. We can apply the Reynolds transport theorem to a 6-dimensional volume being convected through phase space. This tells us that the "conservative derivative" of the probability density of particles is zero:
$\partial_t f + \nabla_R \cdot (Vf) = 0$.
Since $R = (\mathbf{r}, \mathbf{v})$ and $V = (\mathbf{v}, \mathbf{a})$, this says:
$\boxed{\partial_t f + \nabla_\mathbf{r} \cdot (\mathbf{v} f) + \nabla_\mathbf{v} \cdot (\mathbf{a} f) = 0}$ (Liouville equation)

3 Acceleration

Assume that the force on a particle is
$q(\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)) + m\mathbf{g}(\mathbf{r}, t)$,
where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and $\mathbf{g}$ is the gravitational field.

Then $\boxed{\mathbf{a} = \mathbf{a}_p = (\frac{q_p}{m_p}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{g})}$

4 Aside: incompressible flow in phase space.

We can recast this equation in terms of the gradients of f by showing that V is incompressible in phase space, i.e. $\nabla_R \cdot V = 0$. This follows from the following two facts:

(1) $\nabla_\mathbf{r} \cdot \mathbf{v} = 0$, since $\mathbf{r}$ and $\mathbf{v}$ are independent variables.

(2) $\nabla_\mathbf{v} \cdot \mathbf{a} = \frac{q}{m} \frac{\partial}{\partial \mathbf{v}_i} \epsilon_{ijk} \mathbf{v}_j \mathbf{B}_k(\mathbf{r}, t) = 0$

So we can rewrite the Liouville equation as:
$\boxed{\partial_t f + \mathbf{v} \cdot \nabla_\mathbf{r} f + \mathbf{a} \cdot \nabla_\mathbf{v} f = 0}$ (Vlasov equation)

5 Collision Operator

Often it is necessary to take particle collisions into account. The Liouville equation can be modified to incorporate collisions. If f is reasonably smooth, we can handle collisions by time-splitting: we allow the particles to move without colliding for an infinitesimal time interval, and then allow the distribution $f(t, \mathbf{r}, \mathbf{v})$ at each position $\mathbf{r}$ to evolve independently of the distribution at other positions (as if all of space were filled with particles having velocities distributed as they are at the position in question. The nonlinear operator that governs the evolution of a distribution $f_0(t, \mathbf{v})$ is called the *collision operator*, and depends on the physics of the particles. Incoporating the collision operator into the Liouville equation gives the Boltzmann equation:
$\boxed{\partial_t f + \nabla_\mathbf{r} \cdot (\mathbf{v} f) + \nabla_\mathbf{v} \cdot (\mathbf{a} f) = \sum_s C_p^s[f_p, f_s]}$
where $C_p^s[f_p, f_s](\mathbf{r}, \mathbf{v}, t)$ denotes the rate of change of f_p as a result of collisions with particles of species s. Typically we will write C_p^s or merely C^s for $C_p^s[f_p, f_s]$.

6 Moments

Let $M(\mathbf{r}, \mathbf{v}, t)$ be an arbitrary function of phase space.
Define $\int_\mathbf{v} = \int_{\mathbf{v} \in \mathbb{R}^3}$.
Define $\langle M \rangle := \langle M \rangle_p(\mathbf{r}, t) := \frac{\int_\mathbf{v} f_p M}{\int_\mathbf{v} f_p}$ $(\forall M)$.
Let $n := n_p := \int_\mathbf{v} f_p$.
So $\boxed{\int_\mathbf{v} f_p M = n_p \langle M \rangle_p}$.
Let $\boxed{\mathbf{u} := \mathbf{u}_p := \langle \mathbf{v} \rangle_p}$
Let $d_t := d_t^p \partial_t + \mathbf{u} \cdot \nabla$ denote the *convective derivative*.
Let $\delta_t := \delta_t^p := \alpha \mapsto (\partial_t \alpha + \nabla \cdot (\mathbf{u} \alpha))$ denote the *conservative derivative.*
Observe that the following Leibnitz rules hold:

$\delta_t(\alpha\beta) = d_t(\alpha\beta) + (\nabla \cdot \mathbf{u})\alpha\beta$
$= (d_t\alpha)\beta + \alpha(d_t\beta) + (\nabla \cdot \mathbf{u})\alpha\beta$
$= (\delta_t\alpha)\beta + \alpha(d_t\beta)$
$= (d_t\alpha)\beta + \alpha(\delta_t\beta)$.

7 General Moment Calculation.

We wish to compute the zeroth, first, and second velocity moments.
Let $\chi(\mathbf{v})$ be a tensor of monomials in the components of $\mathbf{v}$. We are specifically interested in the cases:

$$\chi(\mathbf{v}) = \begin{cases} 1 & \text{zeroth moment} \\ \mathbf{v} & \text{first moment} \\ v^2 & \text{second moment} \end{cases}$$

Apply the moment-generating operator $\alpha \mapsto \int_\mathbf{v} \alpha\chi$ to the terms of the Boltzmann equation:
$\partial_t \int_\mathbf{v} f\chi + \int_\mathbf{v} \nabla_\mathbf{r}\cdot(\mathbf{v}f)\chi + \int_\mathbf{v} \nabla_\mathbf{v}\cdot(\mathbf{a}f)\chi = \sum_s \int_\mathbf{v} C^s\chi$

These terms simplify as follows:

- (density term) $= \partial_t \int_\mathbf{v} f\chi = \partial_t(n\langle\chi\rangle)$
- (velocity term) $= \int_\mathbf{v} \nabla_\mathbf{r}\cdot(\mathbf{v}f)\chi = \nabla_\mathbf{r}\cdot\int_\mathbf{v}(\mathbf{v}f\chi)$ $= \nabla\cdot(n\langle\mathbf{v}\chi\rangle)$
- (force term) $= \int_\mathbf{v} \nabla_\mathbf{v}\cdot(\mathbf{a}f)\chi$ $= \underbrace{\int_\mathbf{v}(\nabla_\mathbf{v}\cdot(\mathbf{a}f\chi))}_{0 \text{ if } f\to 0 \text{ fast}} - \int_\mathbf{v}(\mathbf{a}f\cdot\nabla_\mathbf{v}\chi)$ $= -n\langle\mathbf{a}\cdot\nabla_\mathbf{v}\chi\rangle$
- (collision term) $= \sum_s \int_\mathbf{v} C^s\chi$

So the general averaged moment of the Boltzmann equation becomes:
$\partial_t(n\langle\chi\rangle) + \nabla\cdot(n\langle\mathbf{v}\chi\rangle) = n\langle\mathbf{a}\cdot\nabla_\mathbf{v}\chi\rangle + \sum_s\int_\mathbf{v} C^s\chi$
Multiply this equation by m and let $\boxed{\rho := mn}$ to get a form that is convenient for expressing conservation laws.
$\boxed{\partial_t(\rho\langle\chi\rangle) + \nabla\cdot(\rho\langle\mathbf{v}\chi\rangle) = \rho\langle\mathbf{a}\cdot\nabla_\mathbf{v}\chi\rangle + \sum_s\int_\mathbf{v} C^s m\chi}$
(Averaged moment Boltzmann equation for momentum and energy balance.)
(Remember to read here $\langle\rangle_p$, n_p, ρ_p, m_p, $\mathbf{a}_p$, and C_p^s.)

8 Collisions and conservation.

The collision operator satisfies the following conservation constraints.

8.1 Conservation of particles. ($\chi = 1$)

Collisions cannot change the number of particles at a given position, so:

$$\int_\mathbf{v} C_p^s = 0$$

This ignores any particle production or destruction processes such as ionization or recombination.

8.2 Conservation of momentum and energy. ($\chi = \mathbf{v}, v^2$)

Let $\boxed{\mathbf{R}^s := \mathbf{R}_p^s := \int_\mathbf{v} C_p^s m_p\mathbf{v}}$ represent the drag force on species p due to collisions with species s.

Let $\boxed{K^s := K_p^s := \frac{1}{2}\int_\mathbf{v} C_p^s m_p v^2}$ represent the energy transfer to species p from species s due to collisions.

Collisions between particles of the same species cannot change the total momentum and energy of that species, so:

$$\mathbf{R}_p^p = 0 \text{ and } K_p^p = 0$$

Collision between different species must conserve the total momentum and energy of the two species, so:

$$\mathbf{R}_p^s + \mathbf{R}_s^p = 0 \text{ and } K_p^s + K_s^p = 0$$

8.3 Relative momentum transfer.

We show that the collisional drag force depends on the particle velocities relative to the average fluid velocity.

Write $\mathbf{v} = \mathbf{u} + \mathbf{c}$, where $\mathbf{c}$ is called the "peculiar" or "thermal" velocity. (i.e. $\boxed{\mathbf{c}_p := \mathbf{v} - \mathbf{u}_p}$.)
Observe that $\langle\mathbf{c}\rangle = 0$.

Observe by conservation of particles that $\boxed{\mathbf{R}_p^s = \int_\mathbf{v} C^s m\mathbf{c}}$.

8.3.1 Form of drag force constitutive relation.

It is often assumed that the drag force is proportional to the difference between the average velocities of the species:
$\mathbf{R}_p^s = m_p n_p \nu_{ps}(\mathbf{u}_s - \mathbf{u}_p)$
where the proportionality constant ν_{ps} is called *the collision frequency for momentum transfer from species s to species p*.
Since $\mathbf{R}_p^s + \mathbf{R}_s^p = 0$, the collision frequencies ν_{ps} and ν_{sp} must satisfy the relation $\rho_p\nu_{ps} = \rho_s\nu_{sp}$.

8.4 Relative energy transfer.

The collision operator can be decomposed into work done by the drag force and heat transfer due to collisions. We use conservation of particles and the definition of the drag force $\mathbf{R}^s$.
The collisional energy transfer is:
$K^s = \frac{1}{2}\int_\mathbf{v} C^s m(\mathbf{u}+\mathbf{c})\cdot(\mathbf{u}+\mathbf{c})$
$= \frac{1}{2}\int_\mathbf{v} C^s m(\mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{c})$
$= \cancel{\int_\mathbf{v} C^s m\frac{1}{2}}u^2 + \mathbf{u}\cdot\int_\mathbf{v} C^s m\mathbf{c} + \int_\mathbf{v} C^s m\frac{1}{2}\mathbf{c}\cdot\mathbf{c}$
So $\boxed{K^s = \mathbf{u}\cdot\mathbf{R}^s + Q^s}$, where $\boxed{Q^s := Q_p^s := \int_\mathbf{v} C^s\frac{1}{2}mc^2}$ denotes direct transfer of thermal energy from species s to species p due to collisions. Note that $\mathbf{R}_p^s\cdot\mathbf{u}$ represent work done by the drag force.

8.4.1 Form of constitutive relation for interspecies thermal energy transfer rate (Q^s).

(Alec's conjecturing.) If the species have zero relative motion, one would naturally posit Newton's law of heat exchange – that heat exchange between species is proportional to the difference in temperature:

$$Q_p^s = k_{ps}\rho_p\rho_s(T_p - T_s)$$

Does such a relation continue to hold if the species have different averaged velocities? Can we use some idea of time splitting here?

9 Zeroth moment: Particle conservation.

Setting $\chi = 1$, we see that the zeroth moment of the Boltzmann equation simply states conservation of particles: $\boxed{\partial_t(\rho) + \nabla \cdot (\rho \mathbf{u}) = 0}$, where $\boxed{\mathbf{u} := \langle \mathbf{v} \rangle}$.

10 First moment: Momentum balance.

Set $\chi = \mathbf{v}$ in the Boltzmann equation to get the momentum balance equation: $\partial_t(\rho\langle\mathbf{v}\rangle) + \nabla\cdot(\rho\langle\mathbf{vv}\rangle) = \rho\langle\mathbf{a}\rangle + \sum_s \mathbf{R}^s$.

Separate microscopic and macroscopic components in the inertial term:
$\langle\mathbf{vv}\rangle = \langle(\mathbf{u}+\mathbf{c})(\mathbf{u}+\mathbf{c})\rangle$
$= \langle\mathbf{uu} + \mathbf{uc} + \mathbf{cu} + \mathbf{cc}\rangle$
$= \mathbf{uu} + \langle\mathbf{cc}\rangle$.

So the momentum equation becomes:
$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{uu}) = -\nabla\cdot(\rho\langle\mathbf{cc}\rangle) + \rho\langle\mathbf{a}\rangle + \sum_s \mathbf{R}^s$.

Make the replacement $\boxed{\mathbf{P} := \rho\langle\mathbf{cc}\rangle = \text{(pressure)}.}$
(Recall that the (gas-dynamic) pressure tensor $\mathbf{P}$ is defined by the property that $\mathbf{n}\cdot\mathbf{P}$ is the surface force per unit area acting on the body (i.e. the diffusive flux of momentum into the body).)

Invoke the *conservative derivative* (for $\mathbf{u}_p$):
$\eth_t := \eth_t^p := \alpha \mapsto \partial_t\alpha + \nabla\cdot(\mathbf{u}_p\alpha)$

So the momentum equation becomes:
$\boxed{\eth_t(\rho\mathbf{u}) = -\nabla\cdot\mathbf{P} + \rho\langle\mathbf{a}\rangle + \sum_s \mathbf{R}^s}$

10.1 Balance of kinetic energy

To obtain a balance law for the macrosopic kinetic energy, dot the momentum balance law with the fluid velocity.
$\eth_t(\rho\mathbf{u})\cdot\mathbf{u} = \rho(d_t\mathbf{u})\cdot\mathbf{u} = \rho d_t(\frac{1}{2}u^2) = \eth_t(\frac{1}{2}\rho u^2)$ So:
$\boxed{\eth_t(\frac{1}{2}\rho u^2) = (-\nabla\cdot\mathbf{P})\cdot\mathbf{u} + \rho\langle\mathbf{a}\rangle\cdot\mathbf{u} + \sum_s \mathbf{R}^s\cdot\mathbf{u}}$

10.2 Form of body force

Recall that $\mathbf{a} = (\frac{q_p}{m_p}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \mathbf{g})$.

Let $\boxed{\mathbf{J}_p := n_p q_p \mathbf{u}_p}$ represent the current.

Let $\boxed{\sigma_p := n_p q_p}$ denote charge density.

So the body force is $\rho\langle\mathbf{a}\rangle = nq_p(\mathbf{E} + \mathbf{u}\times\mathbf{B}) + \rho\mathbf{g}$, i.e. $\boxed{\rho\langle\mathbf{a}\rangle = \sigma\mathbf{E} + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g}}$

So the momentum equation becomes:
$\boxed{\eth_t(\rho\mathbf{u}) = -\nabla\cdot\mathbf{P} + (\sigma\mathbf{E} + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g}) + \sum_s \mathbf{R}^s}$
(Remember to read here $(\rho\mathbf{u})_p$, $\mathbf{P}_p$, σ_p, $\mathbf{J}_p$, and $\mathbf{R}^s_p$.)

11 Second moment: Energy conservation.

Set $\chi = \frac{1}{2}v^2$ in the Boltzmann equation to get conservation of energy:
$\partial_t(\frac{1}{2}\rho\langle v^2\rangle) + \nabla\cdot(\frac{1}{2}\rho\langle\mathbf{v}v^2\rangle) = \rho\langle\mathbf{a}\cdot\mathbf{v}\rangle + \sum_s \int_{\mathbf{v}} C^s \frac{1}{2}mv^2$

To express in macroscopic quantities,
use $\mathbf{v} = \mathbf{u} + \mathbf{c}$ and $\langle\mathbf{c}\rangle = 0$ in each term.
Note that $v^2 = (\mathbf{u}+\mathbf{c})\cdot(\mathbf{u}+\mathbf{c})) = u^2 + 2\mathbf{u}\cdot\mathbf{c} + \mathbf{c}^2$.

The means become:

- $\langle v^2\rangle = \langle(\mathbf{u}+\mathbf{c})\cdot(\mathbf{u}+\mathbf{c})\rangle = \langle u^2 + 2\mathbf{u}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{c}\rangle$
  $= u^2 + \langle c^2\rangle$.
- $\langle\mathbf{v}v^2\rangle$
  $= \langle(\mathbf{u}+\mathbf{c})(u^2 + 2\mathbf{u}\cdot\mathbf{c} + \mathbf{c}^2)\rangle$
  $= \langle(\mathbf{u}u^2 + \cancel{2\mathbf{uu}\cdot\mathbf{c}} + \mathbf{u}c^2 + \cancel{\mathbf{c}u^2} + 2\mathbf{cu}\cdot\mathbf{c} + \mathbf{c}\mathbf{c}^2\rangle$
  $= \mathbf{u}u^2 + \mathbf{u}\langle c^2\rangle + 2\langle\mathbf{cc}\rangle\cdot\mathbf{u} + \langle\mathbf{c}c^2\rangle$
- Claim $\langle\mathbf{a}\cdot\mathbf{v}\rangle = \langle(\frac{q}{m}\mathbf{E} + \mathbf{g})\cdot\mathbf{v}\rangle = \langle\mathbf{a}\rangle\cdot\langle\mathbf{v}\rangle$.
  Indeed, $\langle\mathbf{a}\cdot\mathbf{v}\rangle = \langle(\frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) + \mathbf{g})\cdot\mathbf{v}\rangle$
  $= \langle(\frac{q}{m}(\mathbf{E}+\mathbf{g})\cdot\mathbf{v}\rangle + \cancel{\langle\frac{q}{m}(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\rangle}$
  $= (\frac{q}{m}\mathbf{E} + \mathbf{g})\cdot\langle\mathbf{v}\rangle$.
  Similarly, $\langle\mathbf{a}\rangle\cdot\langle\mathbf{v}\rangle$
  $= \langle(\frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) + \mathbf{g})\rangle\cdot\langle\mathbf{v}\rangle$
  $= (\frac{q}{m}(\mathbf{E}+\langle\mathbf{v}\rangle\times\mathbf{B}) + \mathbf{g})\cdot\langle\mathbf{v}\rangle$
  $= (\frac{q}{m}\mathbf{E} + \mathbf{g})\cdot\langle\mathbf{v}\rangle + \cancel{\frac{q}{m}\langle\mathbf{v}\rangle\times\mathbf{B}\cdot\langle\mathbf{v}\rangle}$, as needed.

  So $\rho\langle\mathbf{a}\cdot\mathbf{v}\rangle = (\rho\langle\mathbf{a}\rangle)\cdot\mathbf{u} = (\sigma\mathbf{E} + \rho\mathbf{g})\cdot\mathbf{u}$

Substituting these values for the averages in the second moment equation above gives the equation:
$\partial_t\left(\frac{1}{2}\rho(u^2 + \langle c^2\rangle)\right) + \nabla\cdot\left(\frac{1}{2}\rho(\mathbf{u}u^2 + \mathbf{u}\langle c^2\rangle)\right)$
$= \rho\langle\mathbf{a}\rangle\cdot\mathbf{u} - \nabla\cdot(\rho\langle\mathbf{cc}\rangle\cdot\mathbf{u}) - \nabla\cdot\left(\frac{1}{2}\rho\langle\mathbf{c}c^2\rangle\right) + \sum_s K^s$.
Make the replacements

$$\mathbf{P} = \rho\langle\mathbf{cc}\rangle = \text{(pressure)},$$

$$\boxed{\begin{array}{llll} \mathbf{q} & := \frac{1}{2}\rho\langle\mathbf{c}c^2\rangle & = & \text{(heat flux per volume), and} \\ e & := \frac{1}{2}\langle c^2\rangle & = & \text{(thermal energy per mass).} \end{array}}$$

Get: $\partial_t\left(\rho(\frac{1}{2}u^2 + e)\right) + \nabla\cdot\left(\rho\mathbf{u}(\frac{1}{2}u^2 + e)\right)$
$= (\rho\langle\mathbf{a}\rangle)\cdot\mathbf{u} - \nabla\cdot(\mathbf{P}\cdot\mathbf{u}) - \nabla\cdot\mathbf{q} + \sum_s K^s$, i.e.

$$\boxed{\begin{array}{lll} \eth_t\left(\frac{1}{2}\rho u^2 + \rho e\right) & = & (\sigma\mathbf{E} + \rho\mathbf{g})\cdot\mathbf{u} - \nabla\cdot(\mathbf{P}\cdot\mathbf{u}) - \nabla\cdot\mathbf{q} \\ & & + \sum_s(\mathbf{R}^s\cdot\mathbf{u} + Q^s) \end{array}}$$

(Remember to read here $\eth_t^p$,$(\frac{1}{2}\rho u^2 + \rho e)_p$, σ_p, ρ_p, $\mathbf{u}_p$, $\mathbf{P}_p$, $\mathbf{q}_p$, $\mathbf{R}^s_p$, and Q^s_p.)

11.1 Balance of thermal energy

Now subtract the balance of macroscopic kinetic energy from the total energy balance to get thermal energy balance:
$\eth_t(\rho e) = -\mathbf{P} : \nabla\mathbf{u} - \nabla\cdot\mathbf{q} + \sum_s Q^s$