Course notes for multivariable integration (§§16.5–6): Surface Integrals
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1 Overview

When we studied integration on curves we began with line integrals, i.e. integration of scalar fields along curves, and then quickly moved on to our real interest: work integrals (and flux integrals) of *vector* fields along oriented curves.

In the same way, in studying integration on surfaces we will begin with integration of scalar fields over surfaces, and then quickly move to our real interest: flux integrals of vector fields over oriented surfaces, which I will refer to as *surface flux integrals* or simply *flux integrals*.

Theoretically, surface flux integrals lay the groundwork to generalize Green's theorem (i.e. the fundamental theorem of calculus) from two to three dimensions. In practice, however, learning to calculate general surface flux integrals is largely independent of learning the generalization of Green's theorem, just as learning to calculate work integrals was largely independent of learning to use potentials and Green's theorem. In a sense, the point of the multivariable versions of the fundamental theorem of calculus is to learn how to convert difficult integrals into easy integrals, and in practice the simplied integral can often be calculated without doing any parametrization.

2 Integral of a scalar field over a surface

The integral of a vector field $f(\mathbf{r})$ over a surface S is denoted $\iint_S f dA$. It is defined by chopping up the surface into little pieces with area dA and summing up the value f on each piece times the area dA. You can think of f as a density per surface area.

In general, to actually calculate a surface integral you must use a parametrization of the surface. A parametrization of a surface is a smooth one-to-one function from a region of the plane to a surface in three-dimensional space. We will typically denote the parametrization by $\mathbf{r}(u, v)$ and say that it maps a region R of the u-v plane (called the domain of the parametrization) to a surface S in three-dimensional space.

Using a parametrization will allow us to calculate a surface integral using an old-fashioned two-dimensional integral over the region R in the u-v plane. To do this, we chop up the region R into small rectangles. The parametrization maps each rectangle to a parallelogram on the surface S. The area of that parallelogram is dA, and we will calculate it using the cross product. (Recall that the area of the parallelogram spanned by two vectors $\mathbf{v}$ and $\mathbf{w}$ is $\|\mathbf{v} \times \mathbf{w}\|$.) Here's the formula:

$$\iint_S f dA = \iint_R f(\mathbf{r}(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u} \right\| du\, dv.$$

In this formula, du and dv are the lengths of the sides of the rectangle, $\frac{\partial \mathbf{r}}{\partial u} du$ and $\frac{\partial \mathbf{r}}{\partial v} dv$ are vectors that span the parallelogram, and $\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u} \| du\, dv$ is the area dA of the parallelogram.

For a random surface these integrals usually cannot be found in closed form, because the magnitude of the cross product involves a square root. The exercises we give are special cases designed so that you can do the integrals.

Exercise. Find the surface area of a sphere of radius R_0. Hint: Parametrize using cylindrical or spherical coordinates. Answer: $4\pi R_0^2$.

3 Flux of a vector field through a surface

When we studied curves in two dimensions we defined the flux of a vector field $\mathbf{F}$ across a curve C with normal vector $\widehat{\mathbf{n}}$ to be the rate of flow across the curve: $\int_C \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds$, where ds denotes infinitesimal arc length.

The flux of a three-dimensional vector field $\mathbf{F}(\mathbf{r})$ across a surface S with unit normal vector $\widehat{\mathbf{n}}$ is defined to be $\iint_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dA$, where dA denotes infinitesimal surface area. (Remark: some people write $d\mathbf{A} := \widehat{\mathbf{n}}\, dA$ and so denote the flux by $\iint_S \mathbf{F} \cdot d\mathbf{A}$.)

To actually calculate a flux integral you generally need to parametrize the surface. Let $\mathbf{r}(u, v)$ be a parametrization of S with domain R. We again chop up R into small rectangles with sides du and dv. Each such rectangle gets mapped to a parallelogram (spanned by $\frac{\partial \mathbf{r}}{\partial u}\,du$ and $\frac{\partial \mathbf{r}}{\partial v}\,dv$). The normal vector $\widehat{\mathbf{n}}$ is perpendicular to this parallelogram, and the area dA is the area of the parallelogram. That is, $\widehat{\mathbf{n}}$ is plus or minus the direction vector of the cross product of the sides, and dA is its magnitude. So $\widehat{\mathbf{n}}\,dA = \pm\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,du\,dv$. We say that the parametrization is **positively oriented with respect to $\widehat{\mathbf{n}}$** if $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is in the same direction as $\widehat{\mathbf{n}}$; otherwise the parametrization is **negatively oriented**. So assuming that $\mathbf{r}(u,v)$ is positively oriented with respect to $\widehat{\mathbf{n}}$,

$$\iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dA = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\,du\,dv.$$

When you calculate a surface integral you must always make sure that the parametrization is positively oriented, i.e., that the cross product of partial derivatives of the parametrization points in the correct direction. Recall that the cross product is anticommutative (i.e. $\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v}$), so if $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}$ points the wrong way you can just reverse the order of u and v to get a parametrization for which the cross product of the partial derivatives points in the proper direction. (Of course you can also just compute the flux with the wrong orientation and then slap a minus sign on your answer.)

Exercise. Find the flux of the vector field $\mathbf{F} = (3y, 3, 4x)$ through the surface parametrized by $\mathbf{r}(u,v) = (u^2, v, uv)$ where $0 \le u \le 1$ and $0 \le v \le 2$. (Assume that $\widehat{\mathbf{n}}$ agrees with the parametrization.) *Solution:* $\frac{\partial \mathbf{r}}{\partial u} = (2u, 0, v)$ and $\frac{\partial \mathbf{r}}{\partial v} = (0, 1, u)$. So $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = (-v, -2u^2, 2u)$. $\mathbf{F}(\mathbf{r}(u,v)) = (3v, 3, 4u^2)$. So $\mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = -3v^2 - 6u^2 + 8u^3$. So $\iint \mathbf{F}\cdot\widehat{\mathbf{n}}\,dA = \int_{v=0}^{2}\int_{u=0}^{1}(-3v^2 - 6u^2 + 8u^3)\,du\,dv = -8$.

4 Orientation (and summary of curve and surface integrals)

4.1 Scalar field integrals. The issue of orientation does not arise when you integrate a scalar field. The line integral

$$\int_C f\,ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\left\|\frac{d\mathbf{r}}{dt}\right\|dt$$

gives you the same value regardless of the direction in which the parametrization traverses the curve, because of the absolute value sign. Similarly, the surface integral

$$\iint_S f\,dA = \iint_R f(\mathbf{r}(u,v))\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial u}\right\|du\,dv$$

gives the same value regardless of which side of the surface the cross product of partial derivatives points toward. In both cases the absolute value eliminates the issue of orientation and tends to makes the integral difficult to calculate.

4.2 Vector field integrals and Orientation. In contrast, whenever you integrate a vector field you must consider the issue of orientation. The work integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t_0}^{t_1}\mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$$

is negated if the parametrization is reversed because $d\mathbf{r} = \frac{d\mathbf{r}}{dt}\,dt$ is negated. Similarly, in two dimensions when you parametrize the line flux integral $\int_C \mathbf{F}\cdot\widehat{\mathbf{n}}\,ds$, the element $\widehat{\mathbf{n}}\,ds$ is the element $d\mathbf{r} = (dx, dy)$ rotated 90 degrees clockwise, $(dy, -dx)$, or counterclockwise, $(-dy, dx)$. You must figure out which way to rotate $d\mathbf{r}$ to point in the same direction as $\widehat{\mathbf{n}}$. So:

$$\int_C \mathbf{F}\cdot\widehat{\mathbf{n}}\,ds = \pm\int_{t_0}^{t_1} F_1\,dy - F_2\,dx$$

where the value of $\pm$ depends on $\widehat{\mathbf{n}}$ and the orientation of the parametrization; it is positive if $\widehat{\mathbf{n}}\cdot(dy, -dx)$ is positive.

Exercise. Find the outward flux of the vector field $\mathbf{F} = (x, 0)$ out of the elliptical region satisfying $x^2 + y^2/4 \le 1$. Answer: 4π.

Likewise, in three dimensions the surface flux integral

$$\iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dA = \pm\iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\,du\,dv,$$

where the value of $\pm$ is positive if $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}$ points in the direction of $\widehat{\mathbf{n}}$, else negative.

Exercise. Find the outward flux of the vector field $\mathbf{F} = (x, y, z)$ through the walls of the solid cylinder which is the solution set to $0 \le z \le 1$, $x^2 + y^2 \le 1$. Answer: 3π.