Course notes for multivariable integration (chapter 16) part 2:

Green's circulation and divergence theorems for integrals in the plane

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1 Overview

This is the second part of a three-part exposition of integral vector calculus. In the first part we studied work integrals along paths and stated that the work integral of the gradient of a potential along an oriented path is its change in value. In this second part we study integrals around closed loops in the plane. In the third section we will generalize this result by studying integrals over arbitrary surfaces in three-dimensional space.

2 Circulation integrals

A **circulation integral** is simply a work integral around a closed curve (i.e. a loop). So, as before, we let C denote an oriented curve with parametrization $\mathbf{r}(t)$, where t runs from t_0 to t_1. To assert that a curve is closed, we add the requirement that $\mathbf{r}(t_0) = \mathbf{r}(t_1)$. We again denote the vector field by $\mathbf{F}(\mathbf{r}) = (M(x,y), N(x,y))$. When we integrate around a closed curve, we draw a circle through the integral to emphasize that the curve is a loop:

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C M\,dx + N\,dy \\ &= \int_{t_0}^{t_1} M\frac{dx}{dt} + N\frac{dy}{dt}\,dt.\end{aligned}$$

We will usually be concerned with *simple* closed curves. A simple closed curve is a closed curve that does not intersect itself. (That is, $\mathbf{r}(t)$ is one-to-one except for the endpoints.) A simple closed curve in the plane is the boundary of a *simply connected region*, i.e a region consisting of only one piece with no holes.

Since the integral goes in a circle, it doesn't actually matter where we start and end, but the *direction* of the integral matters. If we reverse the orientation of the parametrization of the loop the integral is negated. Therefore we usually indicate the direction of orientation of the integral with a counterclockwise or clockwise arrow. Taking $\mathbf{r}(t)$ to be a counterclockwise parametrization,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_{t_0}^{t_1} M\frac{dx}{dt} + N\frac{dy}{dt}\,dt \\ &= -\int_{t_1}^{t_0} M\frac{dx}{dt} + N\frac{dy}{dt}\,dt \\ &= -\oint_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

Actually, nothing in this argument depends on the fact that the curve is closed, and in general we can say that *reversing the direction of parametrization of a curve negates work integrals along it.* Loop integrals in the x-y plane are conventionally counterclockwise (the same direction in which the unit circle is traversed by the standard parametrization $(\cos(t), \sin(t))$.

2.1 Circulation integral of a conservative vector field is zero.

If $\mathbf{F}$ has a potential, i.e., $\mathbf{F} = \nabla\phi$, then the integral is extremely simple:

$$\oint_C \nabla\phi\cdot d\mathbf{r} = \oint_C \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy = \oint_C d\phi = 0,$$

since ϕ has no net change around the loop. Note that the following statements are equivalent:

1. $\mathbf{F}$ is a conservative vector field,
2. $\mathbf{F}$ has a potential,
3. work integrals of $\mathbf{F}$ are path-independent, and
4. circulation integrals of $\mathbf{F}$ are zero.

To see that the last two are equivalent, the difference between the integrals along two different oriented curves C_1 and C_2 from an initial point $\mathbf{r}_0$ to a final point $\mathbf{r}_1$ can be thought of as the integral along C_1 followed by the integral along the *reversal* of C_2: $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} - \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1-C_2}\mathbf{F}\cdot d\mathbf{r}$, where negating a path means reversing its direction and adding two paths means joining them end to end.

2.2 Green's circulation theorem

Even if $\mathbf{F}$ does not have a potential, the fact that the curve is closed will allow us to find the work integral in terms of an integral of a derivative over the region it encloses. This relationship is called Green's theorem, and it is the fundamental theorem of calculus relating loop integrals and area integrals in the plane.

2.3 Derivation of Green's theorem for a rectangle

To discover Green's theorem, consider the simple case of a work integral around a rectangle. Let R be the rectangle $[x_1, x_2] \times [y_1, y_2]$. We denote the boundary of R by ∂R. (This notation is meant to suggest the fact there is a fundamental relationship between boundaries and derivatives.) The work integral of some vector field $\mathbf{F}(\mathbf{r}) = (M(x, y), N(x, y))$ around ∂R is

$$\oint_{\partial R} M\,dx + N\,dy = \oint_{\partial R} M\,dx + \oint_{\partial R} N\,dy.$$

This integral is the sum of the work along each of the four directed line segments that cycle around ∂R. Along the vertical segments dx is zero, and along the horizontal segments dy is zero. Along the vertical segments x is frozen and we can use y as the parameter. Since the path is counterclockwise, y goes from low to high when x is high ($x = x_2$) and from high to low when x is low ($x = x_1$). So:

$$\begin{aligned}
\oint_{\partial R} N\,dy &= \int_{y=y_0}^{y_1} N(x_2, y)\,dy + \int_{y=y_1}^{y_0} N(x_1, y)\,dy \\
&= \int_{y=y_0}^{y_1} N(x_2, y)\,dy - \int_{y=y_0}^{y_1} N(x_1, y)\,dy \\
&= \int_{y=y_0}^{y_1} [N(x_2, y) - N(x_1, y)]\,dy.
\end{aligned}$$

If we freeze the value of y then $N(x, y)$ is a function of x and $N(x_2, y) - N(x_1, y)$ is the difference in the value of this function at two points. But the fundamental theorem of calculus says that *the difference of the value of a function at two points is the integral of its derivative over the interval between the points*, so $[N(x_2, y) - N(x_1, y)] = \int_{x=x_0}^{x_1} \frac{\partial}{\partial x} N(x, y)\,dx$. Therefore,

$$\oint_{\partial R} N\,dy = \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} \frac{\partial}{\partial x} N(x, y)\,dx\,dy.$$

Exercise. Do a similar calculation to show that $\oint_{\partial R} M\,dx = \iint_R -\frac{\partial}{\partial y} M(x, y)\,dy\,dx$. Why is there a minus sign? (Hint A: when you parametrize in the counterclockwise direction, dy is positive when x is high, but dx is positive when y is low. Hint B: you can make use of symmetry to swap x and y in the formula $\oint_{\partial R} N\,dy = \iint \frac{\partial}{\partial x} N(x, y)\,dx\,dy$, but when you do so you must also reverse the direction of circulation to maintain full symmetry:

$$\oint_{\partial R} M\,dx = \iint \frac{\partial}{\partial y} M(x, y)\,dx\,dy.)$$

So we have proved (for a rectangle) the two ingredients of Green's theorems:

1. $\boxed{\oint_{\partial R} N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy}$
2. $\boxed{\oint_{\partial R} M\,dx = \iint_R -\frac{\partial M}{\partial y}\,dy\,dx}$

Putting these two ingredients together, we get **Green's circulation theorem**:

$$\boxed{\oint_{\partial R} M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy}$$

A summary mnemonic: for dy terms in a counterclockwise circulation integral you can insert "canceling" dx and ∂x, but for dx terms *you must change the sign when you insert "canceling" dy and ∂y* (because dx is "negative on the high side").

2.4 Green's theorem for closed curves

Green's theorem actually holds for any simple closed curve. Let's see why. First let's show that it holds for any a simply connected region R that is made up of tiled (adjacent, nonoverlapping) rectangles. Green's theorem holds on each of the little rectangles. Adding up the area integrals for the small rectangle gives an area integral for R. So we just need to show that when you add up the circulation integrals around the rectangles you similarly get the circulation integral around R. The circulation integrals for two adjacent rectangles will traverse their shared edge in opposite directions. But *when you reverse the direction in which a line segment is parametrized, the work along it is negated.* So the net contribution of each shared edge to the total work around all the rectangles is zero. The only edges that are not shared are the edges on the boundary of R. So the total work around all the rectangles in R is the work around the boundary of R.

Now consider a simply connected region with a piecewise smooth boundary. Any such region can be approximated arbitrarily well with a region whose boundary consists of horizontal and vertical line segments. Here's a way to make such an approximation. Break up the boundary into tiny segments with displacement $d\mathbf{r} = (dx, dy)$. Replace each piece with a horizontal displacement dx followed (or preceeded—it doesn't matter) by a vertical displacement dy. The work integral along the displacement $d\mathbf{r}$ is approximated by the work integral along the horizontal and vertical segments. In fact, that's what is suggested by our "differential form" notation $\mathbf{F} \cdot d\mathbf{r} = M\,dx + N\,dy$. But a region whose boundary consists of horizontal and vertical line segments can be tiled with rectangles. Taking the limit of better and better approximations shows that Green's theorem holds for any simply connected region with a piecewise smooth boundary.

2.5 Green's formula for area enclosed by a curve

Most often Green's theorem is used to find a loop integral by integrating derivatives over the the region enclosed by the loop. But we can also use it in the opposite direction. For example, two formulas to calculate area using a work integral are:

$$\iint_R 1\,dx\,dy = \iint_R \frac{\partial x}{\partial x}\,dx\,dy = \oint x\,dy,$$
$$\iint_R 1\,dx\,dy = \iint_R \frac{\partial y}{\partial y}\,dx\,dy = -\oint y\,dx.$$

Averaging these formulas gives the popular formula

$$\iint_R 1\,dx\,dy = \frac{1}{2}\oint x\,dy - y\,dx.$$

When there is a high degree of symmetry between x and y this latter formula sometimes causes things to cancel nicely.

Exercise. Use one of the preceding formulas to calculate the area enclosed by the curve parametrized by $\cos(t), 2\sin(t))$. Answer: 2π.

2.6 Physical interpretation of Green's circulation theorem

We can interpret Green's circulation theorem $\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$ in terms of the curl. The quantity $(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ is referred to as the "two-dimensional" curl. To see why, recall that two-dimensional vectors have a *natural embedding* in three-dimensional space. That is, we can think of a two-dimensional vector as a three-dimensional vector if we add a third component equal to zero. Similarly the two-dimensional vector field $(M(x,y), N(x,y))$ has a natural embedding of in three-space, $\mathbf{F} = (M(x,y), N(x,y), 0)$. Calculating the curl shows that the first two components are zero, but the third component is the usual value of $\widehat{\mathbf{k}} \cdot \nabla \times \mathbf{F} = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$. Also, recall that $d\mathbf{r} = \widehat{\mathbf{T}}\,ds$, where $\widehat{\mathbf{T}}$ is the unit tangent and $ds = \|d\mathbf{r}\|$ is displacement length. So we can rewrite Green's theorem as:

$$\oint_{\partial R} \mathbf{F} \cdot \widehat{\mathbf{T}}\,ds = \iint_R \widehat{\mathbf{k}} \cdot \nabla \times \mathbf{F}.$$

That is, Green's circulation theorem says that *the counterclockwise circulation around ∂R is the integral of the out-of-plane component of the curl over R.* In part three we will generalize this statement slightly to apply to arbitrary surfaces in three-dimensional space and we will christen it *Stokes' circulation theorem.* Applying Stokes' theorem to an infinitesimal region (a region small enough that the curl is approximately contant) shows that *physically the curl is the circulation per area.*

3 Divergence theorem

In two dimensions the divergence theorem and the circulation theorem are two ways of viewing Green's theorem. Just as circulation measures the tendency to circulate around a loop, flux measures the tendency to flow out of a loop.

Flux refers to the (rate of) flow of some kind of stuff across a boundary. Let $\mathbf{v}(\mathbf{r})$ be a vector field which represent the velocity of a substance of density ρ as a function of position in space. Let $d\mathbf{r}$ denote a short line segment in space with length ds and unit normal vector $\widehat{\mathbf{n}}$. The amount of stuff that flows across this line segment in the direction of $\widehat{\mathbf{n}}$ in time dt is $\rho\mathbf{v} \cdot \widehat{\mathbf{n}}\,dt\,ds$, i.e., the amount of stuff per area (ρ) times the area of stuff passing through the line segment. The **flux vector** is defined to be $\mathbf{F} := \rho\mathbf{v}$. So the rate of flow across $d\mathbf{r}$ is $\mathbf{F} \cdot \widehat{\mathbf{n}}\,ds$. The rate of flow across a curve is the integral of the rate of flow across infinitesimal segments. So the flux across curve C toward the side $\widehat{\mathbf{n}}$ points toward is: $\int_C \mathbf{F} \cdot \widehat{\mathbf{n}}\,ds$. Consider a counterclockwise parametrization $\mathbf{r}(t)$ of a simple closed curve in the plane. Rotating $d\mathbf{r} = (dx, dy)$ 90 degrees clockwise gives the outward normal vector $\widehat{\mathbf{n}}\, ds = (dy, -dx)$. Using Green's theorem,

$$\begin{aligned}
\oint_{\partial R} \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds &= \oint_{\partial R} \begin{pmatrix} M \\ N \end{pmatrix} \cdot \begin{pmatrix} dy \\ -dx \end{pmatrix} \\
&= \oint_{\partial R} M\, dy - N\, dx \\
&= \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dx\, dy \\
&= \iint_R \nabla \cdot \mathbf{F}\, dx\, dy
\end{aligned}$$

To understand the physical meaning of the divergence, apply this theorem to a very small test region R of area (ΔA) on which $\nabla \cdot \mathbf{F}$ and ρ are approximately constant. The amount of stuff in this region is approximately $\rho(\Delta A)$. The rate of change of the amount of stuff in this region is $(\Delta A)\frac{\partial \rho}{\partial t} = -\oint_{\partial R} \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds = -\iint_R \nabla \cdot \mathbf{F}\, dx\, dy \approx \nabla \cdot \mathbf{F}\, (\Delta A)$. So *the divergence of the flux of stuff is negative the rate of change of the density:* $\boxed{-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{F}}$.

Exercise (more challenging). Let $\rho(t, \mathbf{r})$ be the density of a fluid as a function of space and time, and let $\mathbf{v}(t, \mathbf{r})$ be the fluid velocity. Let $\mathbf{r}(t)$ denote the path of a particle in the fluid. So $\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}(t))$, $\rho(\mathbf{r}(t))$ is the density along the particle's path, and $\frac{d\rho}{dt} := \frac{d\rho(t,\mathbf{r}(t))}{dt}$ denotes the rate of change of the density along the particle's path. (1) Use the chain rule to show that $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$. (2) Show that $-\frac{1}{\rho}\frac{d\rho}{dt} = \nabla \cdot \mathbf{v}$, i.e., the divergence of the velocity is the negative of the logarithmic rate of change of the density. (Hint: using the product rule, $-\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho$; solve for $\nabla \cdot \mathbf{v}$.)

4 Conclusion

In summary Green's theorem in the plane may be viewed as a Divergence theorem, which states that the outward flux is the integral of the divergence:

$$\begin{aligned}
\oint_{\partial R} \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds &= \iint_R \nabla \cdot \mathbf{F}\, dx\, dy, \text{ i.e.,} \\
\oint_{\partial R} M\, dy - N\, dx &= \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dA
\end{aligned}$$

or it may be viewed as a Circulation theorem, which states that the circulation is the integral of (the perpendicular component of) the curl:

$$\begin{aligned}
\oint_{\partial R} \mathbf{F} \cdot \widehat{\mathbf{T}}\, ds &= \iint_R \widehat{\mathbf{k}} \cdot \nabla \times \mathbf{F}, \text{ i.e.,} \\
\oint_{\partial R} M\, dx + N\, dy &= \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA.
\end{aligned}$$

Exercise. Let $\mathbf{F}(x, y) = (3x - y, x + 3y + y^2)$. Let C be the ellipse $x^2 + y^2/4 = 0$.

- What is a counterclockwise parametrization of the ellipse? Answer: $x(t) = (\cos(t), 2\sin(t))$.
- What is the area of the ellipse? (Hint: the ellipse is a unit circle that has been stretched in each axis by a certain factor.) Answer: 2π.
- Calculate the counterclockwise circulation around the ellipse directly.
- What is (the out-of-plane component of) the curl of $\mathbf{F}$? Answer: 2
- Calculate the counterclockwise circulation around the ellipse using Green's theorem. Answer: 4π.
- Calculate the flux out of the ellipse directly.
- What is the divergence of $\mathbf{F}$? Answer: $6 + 2y$.
- Calculate the flux out of the ellipse using Green's theorem. Answer: 12π.