Notes on integration over areas and volumes (Chapter 15)

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We studied the differential calculus of scalar-valued functions in chapter 14. In chapter 15 we studied the integral calculus of scalar-valued functions. This document discusses integration of continuous real-valued functions of two or three variables. We use $f(x, y)$ to denote a generic function from $\mathbb{R}^2$ to $\mathbb{R}$, and we use $f(x, y, z)$ to denote a generic function from $\mathbb{R}^3$ to $\mathbb{R}$.

1 Definition of integration over multidimensional regions.

The integral of a function $f(\mathbf{r})$ over a two-dimensional region A is written as $\iint_A f\,dA$. It is the sum of the area of each infinitesimal piece dA of A multiplied by the value of the function in each piece. If $f = 1$ then $\iint_R f\,dA$ is simply the area of region A. To approximate the integral computationally we use pieces of finite size, and we call the approximation a *Riemann sum*. The integral can be rigorously defined to be a limit of Riemann sums as the size of the pieces goes to zero:

$$\iint_A f\,dA := \lim_{\operatorname{norm}(\{A_k\})\to 0} \sum_k f(\mathbf{r}_k)\Delta A_k;$$

here $\{A_k\}$ represents a *partition* of the region A into (small) non-overlapping pieces (which we will call *cells*) which cover the region A, each $\mathbf{r}_k$ is a sample point lying in its corresponding region (i.e., $x_k \in A_k$), ΔA_k is the *measure* (i.e. area or volume) of cell A_k, and $\operatorname{norm}(\{A_k\})$ is the *norm of the partition*, i.e. the largest cell radius. (The *radius* of a cell is defined to be the the smallest number that is greater than or equal to the distance between any two points in the cell. We say that two cells are *non-overlapping* if there is no ball (disc) that they both contain.)

The integral of a function $f(x, y, z)$ over a three-dimensional volume V is written $\iiint_V f\,dV$. The definitions are just like the definitions for area.

We often simply write $\iint_A f$ or $\iiint_V f$.

Exercise. Let $f(x, y) = xy$ and let $A = [0, 1]\times[0, 2]$. (A is called the *Cartesian product* of the intervals $[0, 1]$ and $[0, 2]$. A is the rectangle of points (x, y) satisfying the inequalities $0 \le x \le 1$ and $0 \le y \le 2$.) Find $\iint_A f\,dA$ by computing a limit of Riemann sums. Answer:

$$\iint_A f\,dA := \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{2n} f\Big(\frac{i}{n}, \frac{j}{n}\Big)\frac{1}{n^2}$$
$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=0}^{n} i \sum_{j=0}^{2n} j$$
$$= \lim_{n\to\infty} \frac{1}{n^4} \frac{n(n+1)}{2} \frac{2n(2n+1)}{2} = 1.$$

2 Iterated integrals

In practice, to calculate multidimensional integrals we use *iterated integrals*. Recall that a one-dimensional *slice* of a multivariable function is a function obtained by freezing all but one of its variables. Just as we used slices to do multivariable differential calculus using the tools of one-variable differential calculus, we also use slices to do multivariable integral calculus using the tools of one-variable integral calculus. Specifically, we can slice and perform one-dimensional integrals iteratively along each dimension to reduce the dimensions of the integral one dimension at a time until we get a single number that represents the value of the integral.

Suppose that a region R is defined by an *iterated inequality* of the form $x_1 \le x \le x_2$, $y_1(x) \le y \le y_2(x)$. Let $f(x, y)$ represent the density of “stuff” in region R. To find the integral $\iint_R f$ we first integrate slices along y from the lower boundary function to the upper boundary function, regarding x as a constant (so we take a "partial antiderivative" with respect to y). This gives us a function of x:

$$S(x) := \int_R f\,dy := \int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy.$$

$S(x)dx$ represents the amount of stuff in a y-slice of infinitesimal thickness dx. To find the total amount of stuff we then integrate $S(x)dx$ from the lower bound x_1 to the higher bound x_2 (this interval is the one-dimensional shadow of R when it is projected onto the x-axis along the y-slices by extending them to hit the x-axis):

$$\iint_R f = \int_{x=x_1}^{x_2} S(x)\,dx$$
$$= \int_{x=x_1}^{x_2} \left(\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy \right) dx.$$

We can extend this idea to three dimensions. Suppose that R is defined by an *iterated inequality* of the form $x_1 \le x \le x_2$, $y_1(x) \le y \le y_2(x)$, $z_1(x,y) \le z \le z_2(x,y)$. Let $f(x,y,z)$ represent the density of "stuff" in region R. To find the integral $\iiint_R f$ we first integrate slices along z from the lower boundary function to the upper boundary function. This gives us a function $S(x,y)$:

$$S(x,y) := \int_R f dz := \int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz.$$

$S(x,y)\,dx\,dy$ represents the amount of stuff in a z-slice of infinitesimal area $dx\,dy$. To find the total amount of stuff we then integrate $S(x,y)\,dx\,dy$ over the two-dimensional region $x_1 \le x \le x_2$, $y_1(x) \le y \le y_2(x)$ (this region is the two-dimensional shadow of R when it is projected onto the x-y plane along the z-slices):

$$\iiint_R f = \int_{x=x_1}^{x_2} \int_{y=y_1(x)}^{y_2(x)} S(x,y)\,dy\,dx.$$
$$= \int_{x=x_1}^{x_2} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Most regions do not come dressed up as the solution set of an iterated inequality. Usually a region is defined using the equations of the boundaries. To represent a region using iterated inequalities do the following:

1. Choose an order of integration (let's say $dz\,dy\,dx$).
2. Solve the boundary equations for z (the first integration variable).
3. Chop up the region into subregions which can be represented using an iterated inequality. In particular, you will need to define subregions which lie between a lower boundary $z_1(x,y)$ and an upper boundary $z_2(x,y)$. Then you will need to project each subregion onto the x-y plane and repeat this process for its shadow region.

Here are some tips for doing iterated integrals:

- It is often the case that whether an iterated integral is easy or difficult (or even possible in terms of elementary functions) depends on the order of integration that you choose.
- I generally find it much easier to sketch the regions than to try to work directly with the inequalities that define them. Inequalities are hard!
- Pay attention to what is constant and what may vary. If you see something like $I := \int_{x=0}^{2} \int_{y=1}^{2} f(x)g(y)\,dy\,dx$ then $f(x)$ is a constant in the inner integrand, so you can factor it out. Then the inner integral will give a number independent of x. So we can factor it out of the outer integral. So $I = \left(\int_0^2 f(x)\,dx \right) \left(\int_1^2 g(y)\,dy \right)$.

Some people like to write iterated integrals using the notation $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,dy\,dx$, where dx and dy function as closing parentheses for their respective integrals; others like to write something like $\int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy\, f(x,y)$, making it easier to see which variable is being integrated for each integral. I choose to write $\int_{x=x_1}^{x_2} \int_{y=y_1}^{y_2} f(x,y)\,dy\,dx$ in order to accomplish both ends.

3 Change of coordinates

The boundaries are often the most difficult aspect of multidimensional problems. Therefore, it is often convenient to change coordinates and calculate the integral in a set of coordinates in which the boundary equations (or the integrand) take a simple form. When there is rotational symmetry polar, cylindrical, or spherical coordinates are often appropriate. The tricky part about changing coordinates is getting the volume element correct.

3.1 Polar coordinates

Polar coordinate specify the position of a point in the plane in terms of its distance r from the origin and its angle θ measured counterclockwise from the positive x-axis. Right triangle trigonometry shows that

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ r^2 &= x^2 + y^2. \end{aligned}$$

Suppose that the region R is the solution set to the system of iterated inequalities

$$\begin{aligned} \theta_1 \le \theta \le \theta_2, \\ r_1(\theta) \le r \le r_2(\theta). \end{aligned}$$

Consider a fixed value of r and θ. Allowing r to vary by a small increment dr and allowing θ to vary by a small independent increment $d\theta$ traces out an infinitesimal rectangle with sides $(rd\theta)$ and dr. The area of this infinitesimal element of surface area is $r\,dr\,d\theta$. So the integral of $f(x, y)$ over R is expressed in polar coordinates as

$$\iint_R f = \int_{\theta=\theta_1}^{\theta_2} \int_{r=r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Always remember to include the factor of r in the area element!

3.2 Cylindrical coordinates

Cylindrical coordinates (r, θ, z) are simply polar coordinates plus a z axis. They specify the position of a point in space by taking the rectangular (cartesian) coordinates (x, y, z) and replacing the x and y coordinates with the corresponding polar coordinates. Fixing the coordinates and allowing them to vary independently by increments dr, $d\theta$, and dz traces out an infinitesimal box with sides $r\,d\theta$, dr, and dz. The infinitesimal volume element is thus $dz\,r\,dr\,d\theta$.

Exercise. in cylindrical coordinates what is the integral of the function $f(x, y, z)$ over the solution set of the iterated inequalities

$$\begin{aligned} 0 \le r \le r_2(\theta), \\ z_1(r) \le z \le z_2(r)? \end{aligned}$$

Answer:

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{r_2} \int_{z=z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) dz\,r\,dr\,d\theta.$$

3.3 Spherical coordinates

Spherical coordinates (ρ, θ, ϕ) specify the position of a point in space in terms of its distance ρ from the origin, its angle ϕ from the z-axis, and its angle θ from the x-z plane (measured "counterclockwise" about the positive z-axis, i.e. from the x-axis toward the y-axis). Spherical coordinates are obtained from cylindrical coordinates by replacing the variables r and z with the variables ρ and ϕ using a right triangle with sides r, z, and ρ and angle ϕ:

$$\begin{aligned} r &= \rho\sin\phi, \\ z &= \rho\cos\phi, \\ \rho^2 &= r^2 + z^2. \end{aligned}$$

Fixing the coordinates and allowing them to vary independently by increments $d\rho$, $d\theta$, and $d\phi$ traces out an infinitesimal box with sides $d\rho$, $r\,d\theta$, and $\rho\,d\phi$. Since $r = \rho\sin\phi$, the infinitesimal volume element is thus $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.

Exercise. In spherical coordinates, what is the integral of $g(r, \theta, z)$ over the solution set to $0 \le \rho \le \rho_2$? Answer:

$$\int_0^{2\pi} \int_0^{\pi} \int_{\rho_1}^{\rho_2} g(\rho\sin\phi, \theta, \rho\cos\phi)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

3.4 General coordinates

A generic way to look for coordinates in which the boundaries have simple equations is to regard the boundaries as level sets of some function.

For example, suppose that you are asked to integrate the function $f(x, y)$ over the region R defined by the inequalities $1 \le xy \le 9$, $4 \le x/y \le 25$. This region would be messy to deal with in x-y coordinates. We want to write the boundary equations of level sets of functions $u(x, y)$ and $v(x, y)$ such that it is easy to solve for x and y in terms of u and v.

If we define the variables $\tilde{u} := xy$ and $\tilde{v} := x/y$ then $x = \sqrt{\tilde{u}\tilde{v}}$ and $y = \sqrt{\tilde{u}/\tilde{v}}$. Square roots are often nasty to deal with in integrals. If we instead use the definitions $u^2 := xy$ and $v^2 := x/y$, then $x = uv$ and $y = u/v$. Which choice of coordinates is better will depend on what the integrand $f(x, y)$ looks like.

In x-y coordinates the area element is simply $dA = dx\,dy$. To transform the integral we need to determine the area element in u-v coordinates. To this end, pick an arbitrary fixed point $u = u_0$ and $v = v_0$. These coordinates select a corresponding point in the x-y plane with coordinates $x_0 = x(u_0, v_0)$ and $y_0 = y(u_0, v_0)$. If we allow u and v to vary independently over an infinitesimal box

$$\begin{aligned} u_0 &\le u \le u_0 + du_{\max}, \\ v_0 &\le v \le v_0 + dv_{\max}, \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &\le du \le du_{\max}, \\ 0 &\le dv \le dv_{\max}, \end{aligned}$$

then $x(u, v)$ and $y(u, v)$ will trace out an infinitesimal parallelogram in the x-y plane. The area of this parallelogram is the area element dA that we need so that we can write down the integral in u-v coordinates. The sides of this parallelogram are vectors, which we can get from the linear approximation

$$\begin{aligned} x(u_0 + du, v_0 + dv) &\approxeq x_0 + x_u\,du + x_v\,dv, \\ y(u_0 + du, v_0 + dv) &\approxeq y_0 + y_u\,du + y_v\,dv, \end{aligned}$$

where the partial derivatives (e.g. x_u) are evaluated at (u_0, v_0). That is,

$$\begin{aligned} dx &= x_u\,du + x_v\,dv, \\ dy &= y_u\,du + y_v\,dv. \end{aligned}$$

In particular, if we choose one edge of the box, $(du, dv) = (du_{\max}, 0)$, then we get one edge of the parallelogram, $(dx, dy) = \boxed{(x_u, y_u)\,du_{\max}}$; and if we choose the other edge of the box, $(du, dv) = (0, dv_{\max})$, then we get the other edge of the parallelogram, $(dx, dy) = \boxed{(x_v, y_v)\,dv_{\max}}$. Taking the magnitude of the cross product of the two edges of the parallelogram gives its area: $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du_{\max}\,dv_{\max}$, where

$$\frac{\partial(x, y)}{\partial(u, v)} := x_u y_v - x_v y_u$$

is called the *Jacobian determinant*. The magnitude of the Jacobian determinant is the local volume magnification factor of the transformation $T := (u, v) \mapsto (x, y)$. (The sign of the Jacobian determinant is negative if the sides of the parallelogram have the opposite orientation of the sides of the box.)

So in general the integral becomes

$$\iint_Q f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv,$$

where Q is the region in the u-v plane corresponding to the region R in the x-y plane. For our particular example, $\frac{\partial(x,y)}{\partial(u,v)} = -2u/v$, so if we choose $f(x, y) = x$ then the integral becomes

$$\int_{u=1}^{3} \int_{v=2}^{5} uv(2u/v)\,du\,dv = 16.$$

Exercise. Generalize this section to three dimensions (volume integerals). (Hint: the infinitesimal box $du\,dv\,dw$ becomes an infinitesimal parallelpiped whose volume is given by the triple scalar product

$$\begin{bmatrix} x_u\,du \\ y_u\,du \\ z_u\,du \end{bmatrix} \cdot \begin{bmatrix} x_v\,dv \\ y_v\,dv \\ z_v\,dv \end{bmatrix} \times \begin{bmatrix} x_w\,dw \\ y_w\,dw \\ z_w\,dw \end{bmatrix},$$

i.e. the determinant

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}\,du\,dv\,dw = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} du\,dv\,dw.)$$

Exercise. Find the volume of the region in the first octant satisfying the inequalities $4 \le xy \le 9$, $9 \le yz \le 16$, $9 \le zx \le 25$. (Hint: use the variables $u^2 = xy$, $v^2 = yz$, and $w^2 = zx$ and show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = 4$.) Answer: 8.