What is vector calculus good for?

by Alec Johnson, presented to UW Math Club, December 14, 2009

1 Introduction

I want to show how to derive the laws of the universe of classical physics. The main tools are Gauss's divergence theorem and Stokes' theorem.

The basic law of classical physics are the forces of electromagnetism, the force of gravity, and Newton's law of how particles respond to forces. In modern physics Newton's law becomes **Quantum Mechanics** (the theory of "small things"). The theory of electromagnetism becomes **Special Relativity** (the theory of "fast things"). **General Relativity** (the theory of "fast heavy things") reconciles special relativity with the theory of gravity. **Quantum Electrodynamics** (the theory of "fast small things") reconciles quantum mechanics with special relativity. The big physics problem is to find a theory of **Quantum Gravity** (a theory of "fast heavy small things") that reconciles quantum electrodynamics with general relativity. *String Theory* is a proposal. But it's hard to test (unless we can somehow mimic the Big Bang in a laboratory, like some of the big colliders are trying to do).

2 Conventions

Symbol	Meaning
t	time
$\mathbf{r}$	position in space
$r := \\|\mathbf{r}\\|$	distance from origin
$\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$	velocity
V, S	arbitrary test volume, test surface
$\rho = \rho(\mathbf{r}, t)$	density (stuff per volume)
$\mathbf{F} = \mathbf{F}(\mathbf{r}, t)$	flux (flow density) of stuff (e.g. $\rho\mathbf{v}$)
$\sigma = \sigma(\mathbf{r}, t)$	electrical charge density (net charge per volume)
$\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$	electrical current density (e.g. $\sigma\mathbf{v}$)
$\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$	electric field
$\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$	magnetic field
$\mathbf{g} = \mathbf{g}(\mathbf{r}, t)$	gravitational field
∇, $\nabla\cdot$, $\nabla\times$	gradient, divergence, curl
$\nabla^2 := \nabla \cdot \nabla$	Laplacian

3 Background

Gauss's divergence theorem says that the flux out of the boundary of a region equals the integral of the divergence over the interior:

$$\oiint_{\partial V} dA\, \widehat{\mathbf{n}} \cdot \mathbf{F} = \iiint_V dV\, \nabla \cdot \mathbf{F}.$$

I will write this in the more abbreviated form

$$\boxed{\oint \widehat{\mathbf{n}} \cdot \mathbf{F} = \int \nabla \cdot \mathbf{F}},$$

where the region, volume, area elements, and number of integral signs are understood.

Stokes' circulation theorem says that the circulation around the boundary of a surface equals the flux of the curl through the surface:

$$\oint_{\partial S} ds\, \widehat{\tau} \cdot \mathbf{F} = \iint_S dA\, \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F}.$$

I will write this in the more abbreviated form

$$\boxed{\oint \widehat{\tau} \cdot \mathbf{F} = \int \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F}}.$$

4 Conservation and balance laws

Many kinds of "stuff" are conserved, e.g. mass. Gauss's divergence theorem allows us to write a partial differential equation that says that "stuff" is conserved. Suppose that a fluid has velocity $\mathbf{v}(\mathbf{r}, t)$ and stuff density $\rho(\mathbf{r}, t)$. Pick a fixed test region V. At any time t the amount of stuff in V is $\int_V \rho$. The volume of stuff that passes through a piece of boundary is dA (the cross-sectional area) times $\mathbf{v}\cdot\widehat{\mathbf{n}}\, dt$ (the component of the velocity perpendicular to the area element times the infinitesimal time increment). So the amount of stuff flowing across the piece is $\rho\mathbf{v} \cdot \widehat{\mathbf{n}}\, dt\, dA$ (the amount of stuff per volume ρ times the volume of stuff that flows out). The **flux vector** is defined to be $\boxed{\mathbf{F} := \rho\mathbf{v}}$. It is the density of the flow of stuff. Dividing by dt and integrating over the boundary, the rate at which stuff flows out of the region is $\oint_{\partial V} dA\, \widehat{\mathbf{n}} \cdot \mathbf{F}$. If stuff is conserved, then the rate of change of stuff in V is minus the rate at which it flows out:

$$\boxed{\frac{d}{dt}\int_V \rho = -\oint_{\partial V} \widehat{\mathbf{n}} \cdot \mathbf{F}}.$$

This is the integral form of a conservation law. (I use the integral form when I do computational plasma simulation: I chop up space into little cells and try to estimate the rate at which stuff flows across each cell boundary.) Gauss's theorem allows us to turn the boundary integral into the integral of a divergence. Time and space are independent, so you can bring the time derivative inside the integral (writing it as a partial derivative ∂_t). So we can write this as a single integral:

$$\int_V \partial_t \rho + \nabla \cdot \mathbf{F} = 0$$

This must be true for any V. So the integrand must be zero. So the generic **conservation law** is

$$\boxed{\partial_t \rho + \nabla \cdot \mathbf{F} = 0}.$$

5 Balance Laws.

Let's generalize the idea of a conservation law. Suppose that instead of being conserved the amount of stuff produced in an infinitesimal volume element dV in an infinitesimal time dt is $S(\mathbf{r},t)\,dV\,dt$. (S is negative if more stuff is being destroyed than produced.) Then the rate at which stuff is produced in a test volume V is $\int_V S\,dV$. Instead of a conservation law we have the **balance law**

$$\frac{d}{dt}\int_V \rho = -\oint_{\partial V} \widehat{\mathbf{n}}\cdot\mathbf{F} + \int_V S.$$

Writing this as $\int_V$ (something) $= 0$ leads to

$$\boxed{\partial_t\rho + \nabla\cdot\mathbf{F} = S}. \qquad (1)$$

The fundamental equations of physics are balance laws and therefore can be made to look like this.

6 Basic PDEs

The generic balance law (1) is an example of a **partial differential equation** (**PDE**). It cannot be solved unless the system is closed by specifying $\mathbf{F}$ and S in terms of ρ.

There are three main types of equations that result from closed balance laws. They all involve the Laplacian operator $\nabla^2 := \nabla\cdot\nabla = \partial_x^2 + \partial_y^2 + \partial_z^2$:

1. *Parabolic* equations, which describe smearing processes, most importantly the **heat equation**:

   $$\boxed{\partial_t u = \kappa\nabla^2 u}.$$

2. *Hyperbolic* equations, which describe waves, most importantly the **wave equation**:

   $$\boxed{\partial_t^2 u = c^2\nabla^2 u}.$$

   If u only varies with x then this simplifies to

   $$\partial_t^2 u = c^2\partial_x^2 u.$$

   To verify that waves moving at speed c satisfy this equation, plug in $u(x,t) = f(x-ct)$ or $u(x,t) = f(x+ct)$.

3. *Elliptic* equations, which describe steady-state equilibria, most importantly the **Poisson equation**:

   $$\boxed{\nabla^2 u = S},$$

   where S is an arbitrary known function called a *source term.* The Poisson equation is used to find *potentials* (antiderivatives) of vector functions.

The ability to solve these three equations is the main objective of a first course in partial differential equations.

7 Evolution PDEs

7.1 Heat equation. An example of a balance law is the heat equation. In this case "stuff" ρ is heat energy. Assume that the heat flux is proportional to the temperature gradient: $\mathbf{F} = -\kappa\nabla u$, where u is the temperature and κ is a constant called the heat conductivity. Assume that the change in heat energy is the specific heat C_p times the change in temperature u. Then the generic conservation law becomes

$$C_p\partial_t u = \kappa\nabla^2 u.$$

This equation is satisfied by sinusoidal functions that decay exponentially. Fourier solved this problem by showing how to write any function as a sum of sinusoidal functions (called a *Fourier series*).

7.2 Wave equation. Imagine a taught horizontal sheet of rubber. To model transverse waves, let $u(\mathbf{r},t)$ represent small vertical displacement of each point from equilibrium. To derive the wave equation we will write conservation of the vertical component of momentum for a piece V of the sheet and then turn it into a partial differential equation using Gauss's law.

Let $v = \partial_t u$ be the vertical component of velocity. Newton's second law says that the sum of the forces is mass times acceleration, i.e., the force is the rate of change of momentum: $F = ma = \frac{d(mv)}{dt}$. The vertical momentum of a piece of the sheet is $\int_V \rho_0\partial_t u$, where ρ_0 is the mass density per area. For small slopes the vertical force per length on the boundary curve is approximately proportional to the slope in the direction $\widehat{\mathbf{n}}$ perpendicular to the boundary (i.e. the directional derivative $D_{\widehat{\mathbf{n}}}u = \widehat{\mathbf{n}}\cdot u$), so the total force on the boundary is $\int_{\partial V} T_0\widehat{\mathbf{n}}\cdot\nabla u$, where the proportionality constant T_0 signifies the tension in the sheet. So the integral form of the wave equation is

$$\partial_t\int_V \rho_0\partial_t u = \oint_{\partial V} T_0\widehat{\mathbf{n}}\cdot\nabla u, \text{ i.e., } \partial_t^2 u = \frac{T_0}{\rho_0}\nabla^2 u$$

using Gauss's divergence theorem. Comparing this with the generic wave equation, the speed of the waves is evidently $c := \sqrt{T_0/\rho_0}$.

8 Radial sources of flux

Consider a steady stream of stuff emanating radially from a source region (e.g. photons from a lamp or solar wind from the sun). Steady flow says time derivatives are zero, so balance of particles says

$$\oint_{\partial V} \widehat{\mathbf{n}}\cdot\mathbf{F} = \int_V S, \text{ i.e., } \boxed{\nabla\cdot\mathbf{F} = S}.$$

In other words, in steady flow the rate at which stuff leaves a region is the rate at which it is produced inside the region.

8.1 Inverse square attenuation from a point source. Imagine radially symmetric flow from a *point source* (a small spherical source region) at the origin. The rate of flow through a spherical shell centered at the source is flux density times the area of the sphere. Since area is proportional to radius squared, *the flux density must attenuate in proportion to the inverse of the radius squared.* This is the **inverse square law**. So

$$\mathbf{F} \propto \frac{\widehat{\mathbf{r}}}{r^2} = \nabla\left(\frac{-1}{r}\right)$$

(where the proportionality constant is evidently the strength of the source divided by 4π). This is the flux of a point source. Other sources can be approximated as a sum of point sources. So more complicated sources also have a potential.

8.2 Inverse attenuation from a line source. Consider a two-dimensional version. So imagine stuff emanating radially from a long straight wire. The rate of flow through a cylinder centered on the wire is proportional to flux density times the radius of the cylinder, so *the flux density must attenuate in proportion to the inverse of the radius:*

$$\mathbf{F} \propto \frac{\widehat{\mathbf{r}}}{r} = \nabla \ln r,$$

(where the proportionality constant is evidently the strength of the source divided by 2π).

9 Gravitation

By studying Kepler's laws of planetary motion Newton inferred that the force of gravity between two bodies is proportional to Mmr^{-2}, the product of their masses divided by the square of the distance between them. Define the gravitational field $\mathbf{g}$ to be the acceleration of a small test mass m. Newton's law of gravity says that the gravitational field produced by a point mass M is

$$\mathbf{g} = G\frac{M}{r^2}\widehat{\mathbf{r}},$$

where $\mathbf{r}$ denotes displacement from the point mass and G is a proportionality constant called the universal gravitational constant. So *the gravitational field can be interpreted as a flux whose source is the mass density.* (You can imagine "gravitons" streaming from each massive particle. Then $\mathbf{g}$ is proportional to the flux of gravitons.) So

$$\oint_{\partial V} \widehat{\mathbf{n}} \cdot \mathbf{g} = -k_g \int_V \rho \quad \text{i.e.,} \quad \boxed{\nabla \cdot \mathbf{g} = -k_g \rho},$$

where $k_g := 4\pi G$.

9.1 Gravitational Potential. The gravitational field produced by each particle is the gradient of a potential. (The potential function is $k_g M\|\mathbf{r} - \mathbf{r}_0\|^{-1}$.) So the total gravitational field is also the gradient of a potential: $\mathbf{g} = -\nabla\phi$. So Newton's law of gravity can be written as a Poisson equation:

$$\boxed{\nabla^2\phi = k_g\rho}.$$

To solve the Poisson equation sum (integrate) the potentials of all the particles:

$$\phi(\mathbf{r}) = k_g \int_{\mathbf{r}_0} \rho(\mathbf{r}_0)\|\mathbf{r} - \mathbf{r}_0\|^{-1}\, d\mathbf{r}_0.$$

10 Electromagnetism

10.1 Charge conservation. *Current* is the flow of (electrical) charge. When a charge density σ moves with a velocity $\mathbf{v}$ it produces a current density $\mathbf{J} := \sigma\mathbf{v}$. So current is the flux of charge. Charge is conserved. So **charge conservation** says

$$\frac{d}{dt}\int_V \sigma = -\oint_{\partial V} \widehat{\mathbf{n}} \cdot \mathbf{J}, \text{ i.e., } \boxed{\partial_t\sigma + \nabla \cdot \mathbf{J} = 0}.$$

10.2 Gauss's law. People noticed that an electric charge produces a force on other charges that attenuates in proportion to the inverse of the square of the distance between the charges. This is exactly like gravity. Define the electric field $\mathbf{E}$ to be the force per charge on a small test charge. Let σ represent net charge density. Then electric field is a flux whose source is the charge density:

$$\oint_{\partial V} \widehat{\mathbf{n}} \cdot \mathbf{E} = \int_V \sigma/\epsilon_0, \quad \text{i.e.,} \quad \boxed{\nabla \cdot \mathbf{E} = \sigma/\epsilon_0},$$

where ϵ_0 is a proportionality constant called the *permittivity of free space.*

10.3 No magnetic monopoles. A small rod with balancing positive and negative electric charges on the two ends is called an *electric dipole.* Electric dipoles line up with the electric field.

You can indicate a field with *field lines.* Gravitational field lines radiate from matter and go to infinity. The density and direction of field lines indicate the flow of "gravitons", i.e. the strength and direction of the gravitational field. Electric field lines radiate from positive charges, move parallel to the electric field, and end at negative charges.

Magnetized iron filings in the presence of a magnet orient themselves along lines. We infer the existence of a magnetic field B. We can think of the filings as "magnetic dipoles" with positive and negative "magnetic charges" at each end. But magnetic field lines generally don't start or end (they often go in circles) the way they would if magnetic charges ("monopoles") existed. The number of field lines entering and exiting any volume is the same, i.e., the net flux of the magnetic field through any boundary is zero, because there are **no magnetic monopoles**:

$$\oint_{\partial V} \widehat{\mathbf{n}} \cdot \mathbf{B} = 0 \quad \text{i.e.,} \quad \boxed{\nabla \cdot \mathbf{B} = 0}.$$

10.4 Faraday's law. A changing magnetic flux through a wire loop produces an electrical current around the loop. Faraday knew that current is produced by an electric field, so he inferred that *the circulation of the electric field is minus the rate of change of flux of the magnetic field:*

$$\oint_{\partial S} \widehat{\tau} \cdot \mathbf{E} = -\frac{d}{dt} \iint_S \widehat{\mathbf{n}} \cdot \mathbf{B} \text{ i.e.,} \boxed{\nabla \times \mathbf{E} = -\partial_t \mathbf{B}}.$$

10.5 Ampere's law. Steady current flowing through a wire produces a magnetic field that circulates around the wire. The strength of this magnetic field is inversely proportional to the distance from the wire. This means that current density is a source of circulation of magnetic field. Ampere's law says *the circulation of the magnetic field around a loop is proportional to the flux of the current through the loop*:

$$\oint_{\partial S} \widehat{\tau} \cdot \mathbf{B} = \mu_0 \iint_S \widehat{\mathbf{n}} \cdot \mathbf{J}, \quad \text{i.e.,} \quad \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}},$$

where μ_0 is a proportionality constant called the *permeability of free space.*

10.6 Maxwell's fix to Ampere's law. Is current density the *only* source of circulation of magnetic field? Take the divergence of Ampere's law. Recall that the divergence of the curl is zero, so we just get $\nabla \cdot \mathbf{J} = 0$. But charge conservation says $\partial_t \sigma + \nabla \cdot \mathbf{J} = 0$. So Ampere implies steady charge density, $\partial_t \sigma = 0$. Maxwell added a correction C to Ampere's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + C$. To determine what C should be, take the divergence and use charge conservation and Gauss's law: $\nabla \cdot C = -\mu_0 \nabla \cdot \mathbf{J} = \mu_0 \partial_t \sigma = \mu_0 \epsilon_0 \partial_t \nabla \cdot \mathbf{E}$. This will hold if we define $C = \mu_0 \epsilon_0 \mathbf{E}$, giving the **Maxwell-Ampere law**

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial_t \mathbf{E}}.$$

You can derive this same result using the integral formulation of the equations of electromagnetism. Let V be an arbitrary volume and let $S = \partial V$. Then the bounday of S is empty. Add an unknown correction to the right hand side of Ampere's law and apply Ampere to $S = \partial V$. So the correction equals the flux of $-\mu_0 \mathbf{J}$. But charge conservation says that the flux of $-\mathbf{J}$ is the rate of change of the total charge. Gauss says that the total charge is the flux of $\epsilon_0 \mathbf{E}$. So the correction equals the flux of $\mu_0 \epsilon_0 \partial_t \mathbf{E}$. Assuming that this correction holds for any surface (not just boundaries of volumes) gives us the integral form of the Maxwell-Ampere law:

$$\oint_{\partial S} \widehat{\tau} \cdot \mathbf{B} = \mu_0 \iint_S \widehat{\mathbf{n}} \cdot \mathbf{J} + \mu_0 \epsilon_0 \partial_t \iint_S \widehat{\mathbf{n}} \cdot \mathbf{E}.$$

10.7 Lorentz force law. An electrical charge q experiences a force in the presence of an electromagnetic field. The force of the electric field is by definition $q\mathbf{E}$. Magnetic fields also exert a force on a charge. People noticed that a wire carrying a current through a magnetic field feels a force perpendicular both to the wire and to the magnetic field. This lead to the Lorentz force law for the total electromagnetic force on a charge moving with velocity $\mathbf{v}$:

$$\boxed{(\text{force}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}.$$

11 Light waves

Faraday's law and the Maxwell-Ampere law are coupled equations for the evolution of the electric and magnetic field. For simplicity ignore current: $\mathbf{J} = 0$. Take the time derivative of Faraday's equation, eliminate $\mathbf{E}$ using Maxwell-Ampere, and use $\nabla \cdot \mathbf{B} = 0$ and the vector identity $\nabla \times \nabla \times \mathbf{B} = \nabla \nabla \cdot \mathbf{B} - \nabla \cdot \nabla \mathbf{B}$ to get

$$\partial_t^2 \mathbf{B} = c^2 \nabla^2 \mathbf{B},$$

where $c := \sqrt{1/(\epsilon_0 \mu_0)}$. This is a wave equation for each component of the magnetic field. It says that electromagnetic waves propagate at speed c, the speed of light.

The electric field also obeys a wave equation. Assume also that $\sigma = 0$. Then Maxwell's equations are symmetric between $\mathbf{E}$ and $\mathbf{B}$ and we can conclude that

$$\partial_t^2 \mathbf{E} = c^2 \nabla^2 \mathbf{E}.$$

Lorentz and later Einstein developed the equations of special relativity by assuming that the equations of electromagnetism hold in any reference frame. Einstein realized the true physical significance of assuming that the speed of light is the same in every reference frame.