Course notes for multivariable integration (chapter 16): Curves, Work Integrals, and Potentials
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1 Overview

Recall that multivariable calculus consists of four topics:

1. **Curves**: the calculus of functions from $\mathbb{R}$ to $\mathbb{R}^n$ (think $\mathbf{r}(t)$),
2. **Surfaces**: the differential calculus of *scalar-valued functions of multiple variables*, i.e. function from $\mathbb{R}^n$ to $\mathbb{R}$ (think ∇f).
3. **Volumes**: the integral calculus of scalar-valued functions over regions of space (think $\iint_A f$ or $\iiint_V f$).
4. **Vector Fields**: the calculus of *vector-valued functions of multiple variables*, i.e. function from $\mathbb{R}^n$ to $\mathbb{R}^m$ (think $\mathbf{F}(\mathbf{r})$).

Scalar-valued functions of multiple variables are called *scalar fields*. Scalar fields are used to represent things such as a temperature or density as a function of position in space. Vector-valued functions of multiple variables are called *vector fields*. A **vector field** is a function $\mathbf{F}(\mathbf{r})$ that assigns a vector $\mathbf{F}$ to every point $\mathbf{r}$ in space. Vector fields are used to represent things such as the force of gravity or the velocity of a fluid as a function of position in space. To graph a vector field $\mathbf{F}(\mathbf{r})$ from $\mathbb{R}^2$ to $\mathbb{R}^2$, pick an array of points in the plane and at the position $\mathbf{r}$ of each point in the plane plot the vector $\mathbf{F}(\mathbf{r})$ with its tail positioned at $\mathbf{r}$. The components of $\mathbf{F}$ are scalar fields. Common names for them are $\mathbf{F} = (M, N, P) = (F_1, F_2, F_3)$. (We skip the letter O because on the blackboard it looks too much like 0.)

Integral vector calculus is concerned with the integration of vector fields over curves and surfaces. Curves and surfaces are examples of *manifolds*. A manifold is *smooth*—that is, locally (i.e. on a small scale) a manifold looks flat. Curves locally look like straight lines; curves are also called *1-dimensional manifolds*. Surfaces locally look like planes; surfaces are also called *2-dimensional manifolds*. In a course in *differential manifolds* you would study how to do integration on higher-dimensional manifolds.

Just as there are three kinds of vector multiplication (multiplication by a scalar, the scalar (dot) product, and the vector (cross) product), for vector calculus there are three kinds of derivatives (the gradient, the divergence, and the curl), three kinds of integrals (the work integral, the flux integral, and the circulation integral), and three versions of the fundamental theorem of calculus (work of a potential gradient equals potential difference, Gauss's divergence theorem, and Stokes' circulation theorem).

The multivariable versions of the fundamental theorem of calculus are the key that allows you to convert statements of the basic laws of physics into *partial differential equations* that you can solve using mathematics or computer simulation. For example, Gauss's theorem make it easy to take statements like "the mass (or momentum or energy) is conserved" and turn them into the equations of gas dynamics or elastodynamics. The fundamental theorems of calculus also allow you to write Maxwell's equations of electromagnetism as partial differential equations.

The motivation and application of the calculus of vector fields tends to be backwards from the calculus of scalar functions. When you studied the calculus of scalar functions you studied derivatives and then studied integrals, and you used the fundamental theorem of calculus to turn the integral of a derivative over an interval into a statement about the values of the antiderivative on the boundary of the interval. But with multivariable calculus it is more natural to first study integrals and then study differentiation, and we usually use the fundamental theorems of calculus to turn an integral over the boundary of a region into an integral of the derivative over the interior of the region.

2 Curves

To discuss curves we will use notation and language similar to what we used in chapter 13. You may find it helpful to review the first five sections of my *Notes on Curves (Chapter 13)*.

Recall (from chapter 13) that a curve C is the 1-dimensional shape traced out by a vector-valued function $\mathbf{r}(t)$. A parametrization determines a direction in which the curve is traced, called the *orientation* of the curve. A curve with a direction of travel is called an *oriented curve*. When we integrate vector fields we will integrate over oriented curves.

Let $\mathbf{r}(t)$ trace out the curve C as "time" t runs from initial time t_0 to final time t_1. So C is an oriented curve in space with beginning point $\mathbf{r}_0 := \mathbf{r}(t_0)$ and endpoint $\mathbf{r}_1 := \mathbf{r}(t_1)$.

As in chapter 13, it is useful to define an arc-length parametrization of the curve. So we let the quantity s denote arc length (measured from the initial point $\mathbf{r}_0$) along C. As in chapter 13, we use "quantity notation" to describe curves. So $s(t)$ denotes arc length as a function of time, $\frac{ds}{dt}$ denotes speed, $\frac{d\mathbf{r}}{dt}$ denotes velocity, and $\frac{d\mathbf{r}}{ds}$ is the unit tangent vector. Note that $s_0 := s(t = t_0) = 0$ and $s_1 := s(t = t_1)$ is the length of the curve.

To define an integral along the curve parametrized by $\mathbf{r}(t)$ we break it up into little pieces $d\mathbf{r}$ of length $ds = \|d\mathbf{r}\|$.

3 Line integrals

A line integral is the integral of a scalar field over a curve. We will not do much with line integrals, but they provide a pedagogical stepping stone to integrals of vector fields. The **line integral** of the scalar function $f(\mathbf{r})$ along the curve C parametrized by $\mathbf{r}(t)$ as t runs from t_0 to t_1 is

$$\int_C f\,ds = \int_{t=t_0}^{t_1} f\,\frac{ds}{dt}\,dt = \int_{t=t_0}^{t_1} f\left\|\frac{d\mathbf{r}}{dt}\right\|dt.$$

So we *define* the line integral using an arc-length parametrization, $\int_{s=s_0}^{s_1} f\,ds$, but the fundamental theorem of calculus shows us that we can calculate it using any parametrization $\mathbf{r}(t)$ with the formula $\int_C f\,ds = \int_{t=t_0}^{t_1} f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt$. *Remark:* If we let $f(\mathbf{r}) = 1$, then the line integral becomes $\int_C ds = \int_{t=t_0}^{t_1} \|\mathbf{r}'(t)\|\,dt$, i.e., the length of the curve. We studied this special case of line integrals in chapter 13.

A simple example. Find the line integral of $f(x, y) = xy$ along the curve C parametrized by $\mathbf{r}(t) = (2\cos t, 2\sin t)$ as t runs from 0 to $\pi/2$. *Solution:*

$$\int_C f\,ds = \int_{t=0}^{\pi/2} 4\cos(t)\sin(t)\sqrt{4\cos^2 t + 4\sin^2 t}\,dt$$
$$= \int_{t=0}^{\pi/2} 8\cos(t)\sin(t)\,dt = \int_{t=0}^{\pi/2} 4\sin(2t)\,dt = [2\cos(2t)]_{\pi/2}^{0} = 4.$$

A more interesting example. (May be skipped on a first reading.) Find the line integral of $f(r, \theta) = 1/r$ along the curve C which is the part of the "unit hyperbola" $x^2 - y^2 = 1$ that connects the points $(1, 0)$ and an arbitrary second point in the first quadrant. *Solution:* Parametrize the curve using $\mathbf{r}(t) = (\cosh(t), \sinh(t))$, where t runs from 0 to an arbitrary time t_{final}. (Recall that cosh and sinh are defined to be the even and odd parts of the exponential function: $\cosh(t) := \frac{e^t+e^{-t}}{2}$ and $\sinh(t) := \frac{e^t-e^{-t}}{2}$. From these definitions you can derive differentiation rules such as $\cosh' = \sinh$ and $\sinh' = \cosh$ and hyperbolic trigonometric identities such as $\cosh^2(t) - \sinh^2(t) = 1$; this identity shows that this curve really does lie on the hyperbola.) So

$$\int_C f\, ds = \int_{t=0}^{t_{\text{final}}} \frac{1}{\sqrt{\cosh^2(t) + \sinh^2(t)}} \sqrt{\cosh^2(t) + \sinh^2(t)}\, dt = \int_{t=0}^{t_{\text{final}}} 1\, dt = t_{\text{final}}.$$

We remark (1) that this same result holds if your replace the hyperbola with the unit circle and replace cosh and sinh with cos and sin and (2) that this problem shows that you can define $(\cosh t, \sinh t)$ to be the coordinates of the point $\mathbf{r}_{\text{final}}$ on the unit hyperbola such that the line integral of $1/r$ from $(1, 0)$ to $\mathbf{r}_{\text{final}}$ along the hyperbola is t, just as $(\cos t, \sin t)$ can be defined to be the coordinates of the point $\mathbf{r}_{\text{final}}$ on the unit circle such that the arc length, i.e. the line integral of 1 (or $1/r$) from $(1, 0)$ to $\mathbf{r}_{\text{final}}$ along the unit circle is t. (This problem shows why cosh and sinh are called hyperbolic trigonometric functions.)

4 Work integrals

Just as a line integral allows us to integrate a scalar field $f(\mathbf{r})$ along a curve, a work integral allows us to integrate a vector field $\mathbf{F}(\mathbf{r})$ along a curve.

Recall from elementary physics that when a constant force $\mathbf{F}$ is applied to an object as it moves through a displacement $d\mathbf{r}$ the work performed is the magnitude of the force times the component of the distance in the direction of the force: $dW = \|\mathbf{F}\| \cdot \|d\mathbf{r}\| \cos\theta$, where θ is the angle between the force vector and the displacement vector. (We remark that this is the same as the magnitude of the displacement times the component of the force in the direction of the displacment.) But this is just the dot product:

$$dW = \mathbf{F} \cdot d\mathbf{r}.$$

Now imagine that the force varies from point to point. (For example, the force vector that gravity exerts on an object varies depending on its position relative to Earth.) Imagine that the object moves in the presence of this force along the curved path C parametrized by $\mathbf{r}(t)$ as t runs from t_0 to t_1. To define the work over this path we break it up into infinitesimal straight-line displacements $d\mathbf{r}$ along which $F(\mathbf{r})$ is approximately constant and sum up the work dW on each piece:

$$W = \int_C dW = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=t_0}^{t_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

Writing this out in coordinates:

$$W = \int_C (M, N, P) \cdot (dx, dy, dz) = \int_C M\, dx + N\, dy + P\, dz = \int_{t=t_0}^{t_1} M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\, dt.$$

The expression $\int_C \mathbf{F} \cdot d\mathbf{r}$, i.e., $\int_C M\, dx + N\, dy + P\, dz$, is called the *differential form* of the work integral. The differential form is useful for deriving and proving general theorems, but in general to actually calculate a work integral you need to use a parametrization.

Trivial theoretical exercise. Use the chain rule to show that the value of a work integral remains the same when you change from one parametrization to another. (Hint: it is enough to compare to an arc-length parametrization.)

Example. What is the work performed by the vector field $\mathbf{F}(\mathbf{r}) = (2, 3y)$ along the path $y = x^2$ from the point $(0, 0)$ to the point $(1, 1)$? *Solution:* Parametrize the path C using $\mathbf{r}(t) = (t, t^2)$ as t goes from 0 to 1. Then the work is

$$W = \int_C 2\, dx + 3y\, dy = \int_{t=0}^{1} 2\frac{dx}{dt} + 3y\frac{dy}{dt}\, dt = \int_{t=0}^{1} 2 + (3t^2)2t\, dt = 4.$$

5 Gradient

Recall (e.g. from section 8 of my *Notes on Differential Calculus of Surfaces (Chapter 14)*) that the gradient of a scalar function f is defined to be the vector of partial derivatives of f:

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

We will think of the symbols $\frac{d}{dt}$, $\frac{\partial}{\partial x}$, and ∇ as objects in their own right. The derivative $\frac{d}{dt}$ is an example of an *operator*. An **operator** is a "meta-function": it is a function which takes a function as its input and returns a function as its output. The ordinary derivative is an operator which acts on functions of one variable. The partial derivative $\frac{\partial}{\partial x}$ is an operator which acts on a scalar field and returns a scalar field. We will frequently use the more compact notation $d_t := \frac{d}{dt}$ and $\partial_x := \frac{\partial}{\partial x}$ for these operators. We define the **gradient** to be the vector of partial differentiation operators: $\nabla := (\partial_x, \partial_y, \partial_z)$. The gradient is an operator which acts on a scalar field and returns a vector field:

$$\nabla f = (\partial_x, \partial_y, \partial_z) f = (\partial_x f, \partial_y f, \partial_z f).$$

We can use the chain rule to express the rate of change of a function along a parametrized path in terms of the gradient:

$$\frac{d}{dt} f(\mathbf{r}(t)) = \frac{d}{dt} f(x(t), y(t), z(t)) = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z} = \frac{d\mathbf{r}}{dt} \cdot \nabla f.$$

(In the notation of differentials, $df = d\mathbf{r} \cdot \nabla f$.)

6 Potential functions

You may recall from elementary physics that the work performed by the gravitational force when an object moves from one position $\mathbf{r}_0$ to another position $\mathbf{r}_1$ equals minus the change in gravitational potential energy. That is, there is a scalar field $f(\mathbf{r})$ (called by *gravitational potential* such that $\int_C \mathbf{F} \cdot d\mathbf{r} = -(f(\mathbf{r}_1) - f(\mathbf{r}_0))$, where $\mathbf{r}_0$ and $\mathbf{r}_1$ are the initial and final points of the oriented curve C. This says that the work of gravity depends only on the initial and final position—it is completely independent of the path the object takes from $\mathbf{r}_0$ to $\mathbf{r}_1$.

If such a potential function exists, how can we find it? Well, suppose that such a function exists. Let's see what happens when we move a short distance along one of the axes, say the x-axis. Then $dW = M\,dx = -df$. Dividing by dx, we see that $M = -\frac{\partial f}{\partial x}$. The same argument works if we move along the other axes, so we can conclude that $(M, N, P) = -(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z},)$. That is, $\mathbf{F} = -\nabla f = \nabla(-f)$. We call f a *(physicist's) potential* for F. The scalar field $\phi := -f$ is a *(mathematician's) potential* for F. That is, mathematicians would say that the scalar field ϕ is a **potential function** for the vector field $\mathbf{F}$ if $\mathbf{F} = \nabla\phi$. We will adopt the convention of mathematicians.

Suppose that ϕ is a potential for $\mathbf{F}$. We show that work is independent of of path using the chain rule:

$$W = \int_{t=t_0}^{t_1} \nabla\phi(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{t=t_0}^{t_1} \frac{d\phi(\mathbf{r}(t))}{dt} d\mathbf{r} = [\phi(\mathbf{r}(t))]_{t=t_0}^{t_1} = \phi(\mathbf{r}_1) - \phi(\mathbf{r}_0).$$

Once you are comfortable with differentials you can discard the crutch of a parametrization, and you can write that proof more compactly as:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla\phi \cdot d\mathbf{r} = \int_C d\phi = \phi(\mathbf{r}_1) - \phi(\mathbf{r}_0).$$

This is the first of three versions of the fundamental theorem of calculus for vector fields. It says that the work of the gradient of a function is determined by its value on the boundary of the curve:

$$\boxed{\int_C \nabla\phi \cdot d\mathbf{r} = \int_C d\phi = \phi(\mathbf{r}_1) - \phi(\mathbf{r}_0) \quad \text{where } C \text{ starts at } \mathbf{r}_0 \text{ and ends at } \mathbf{r}_1.}$$

This involves the gradient operator and work integrals. The other two versions of the fundamental theorem of calculus involve surface or area integrals.

6.1 Curl: testing for a potential

How do you know whether a given vector field $\mathbf{F}$ has a potential function? The mathematician's approach to such a question is to suppose that a potential exists and study the consequences. If a potential does not exist then we can expect to arrive at a contradiction, but if a potential does exist then we can expect to discover it. Suppose that $\mathbf{F} = \nabla\phi$, i.e., $(M, N, P) = (\partial_x\phi, \partial_y\phi, \partial_z\phi)$. Then $\partial_y M = \partial_y\partial_x\phi = \partial_x\partial_y\phi = \partial_x N$, i.e., $\partial_x N - \partial_y M = 0$. Rotating components, we have

$$\begin{aligned}
&\partial_x N - \partial_y M = 0,\\
&\partial_y P - \partial_z N = 0,\\
&\partial_z M - \partial_x P = 0.
\end{aligned}$$

So $\mathbf{F}$ will have a potential only if these three quantities are zero. These three quantities are the components of a vector called the *curl* of $\mathbf{F}$. The **curl** of $\mathbf{F}$ is formally defined to be the "cross product" of the vector of partial derivative operators with $\mathbf{F}$. Calculating this out:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} \times \begin{bmatrix}M\\ N\\ P\end{bmatrix} = \begin{bmatrix}\partial_y P - \partial_z N\\ \partial_z M - \partial_x P\\ \partial_x N - \partial_y M\end{bmatrix}.$$

So if a potential function exists then the curl must be zero. And in fact, if the curl is zero everywhere then a potential exists (as we will be able to see once we have proved Green's theorem for a rectangle).

The curl indicates the tendency of a vector field to rotate. When we study Stokes' theorem we will discover the precise physical meaning of the curl.

Exercise. Suppose that $\mathbf{F} = (M(x, y), N(x, y))$ is a vector field in the plane. Show that if $\mathbf{F} = (\partial_x\phi, \partial_y\phi)$ then $\mathbf{F}$ must satisfy $\partial_x N - \partial_y M = 0$.

Exercise. Show that the vector field $\mathbf{F} = (-y, x)$ does not have a potential.

6.2 Finding a potential

Once you know that a potential exists, how do you find it? We illustrate the general technique with an example. Suppose

$$\mathbf{F}(\mathbf{r}) = \begin{bmatrix} e^{x+z}\\ \cos y\\ xe^{x+z} + 1\end{bmatrix}.$$

Exercise: Verify that $\mathbf{F}$ *has a potential by showing that* $\nabla \times \mathbf{F} = 0$.

We now proceed to find a potential. Let ϕ be a potential. Then

$$\begin{aligned}
\partial_x\phi &= e^{x+z},\\
\partial_y\phi &= \cos y,\\
\partial_z\phi &= xe^{x+z} + 1.
\end{aligned}$$

Just as the partial derivative is defined using the ordinary derivative by freezing all variables except for one, the partial antiderivative is defined using the ordinary antiderivative by freezing all variables except for one. Antidifferentiating with respect to x shows us that

$$\phi = xe^{x+z} + C(y, z),$$

where the constant of integration C depends on y and z because ϕ depends on all three variables and we are performing a separate antidifferentiation for every value of y and z. Differentiating this equation with respect to y gives:

$$\partial_y \phi = \partial_y C(y, z) = \cos y$$

Antidifferentiating with respect to y then gives

$$C(y, z) = \sin y + \widetilde{C}(z).$$

($\widetilde{C}$ does not depend on x because nothing in the equation depends on x.) So now we can write

$$\phi = xe^{x+z} + \sin y + \widetilde{C}(z)$$

Differentiating with respect to z gives

$$\partial_z \phi = xe^{x+z} \partial_z \widetilde{C}(z) = xe^{x+z} + 1.$$

Canceling and antidifferentiating with respect to z gives

$$\partial_z \widetilde{C}(z) = z + C_0,$$

So the potential function must be

$$\phi = xe^{x+z} + \sin y + z + C_0.$$

Exercise: Verify that this answer is correct, i.e., $\nabla \phi = \mathbf{F}$.

Exercise. Suppose that ϕ and φ are two potentials for $\mathbf{F}$, i.e., $\mathbf{F} = \nabla \phi$ and $\mathbf{F} = \nabla \varphi$. Show that $\phi = \varphi + C_1$, where C_1 is a constant (independent of x, y, and z). (Hint: What is $\nabla(\phi - \varphi)$?)

Exercise. Find a partner. Pick a secret potential function $\phi(x, y)$ and calculate its gradient $\nabla \phi$. Then make up another vector field $\mathbf{F}(x, y)$. Your partner does the same thing. Now trade vector fields. Can you determine which vector field came from a potential? What is the potential?

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7 Divergence

The divergence has nothing to do with work integrals, but I include it in this section for completeness and since the webwork expects you to know it.

I mentioned that vector fields involve three kinds of derivatives. We have seen the gradient, which takes a scalar field and returns a vector field, and the curl, which takes a vector field and returns a vector field. The third kind of derivative is the *divergence*, which takes a vector field and returns a scalar field. The **divergence** of a vector field $\mathbf{F}(\mathbf{r})$ is formally defined to be the dot product of the vector of partial derivative operators with the vector field:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = (\partial_x, \partial_y, \partial_z) \cdot (M, N, P) = \partial_x M + \partial_y N + \partial_z P.$$

Exercise. Calculate the divergence of the vector field $\mathbf{F}(x, y) = (x, xy)$. Answer: $1 + x$.

Exercise. Show that divergence of the curl is zero. Show that the curl of the gradient is zero.