Derivation of Navier-Stokes

by Alec Johnson, May 26, 2006

1 Derivation of Conservation Laws

1.1 Context and Conventions

By default quantities are functions of space $\mathbf{x}$ and time t.

Let $\mathbf{u}$ be the velocity field (which is convecting the continuum).

Let α, β, and $\mathbf{q}$ stand for arbitrary (convected) quantities.

Let $U(t)$ stand for an arbitrary convected region (volume element). ($U(t)$ is simply connected with smooth boundary.)

Let ∂U denote the boundary of the region U.

Let $\int := \int_{U(t)}$, and let $\oint := \int_{\partial U(t)}$, i.e. the default domain of integration is the arbitrary convected volume element.

Let $\mathbf{n}$ denote the outward unit normal to ∂U.

1.2 Kinetics Calculus

Definitions.

Let $\partial_t := \frac{\partial}{\partial t}$.

Let $\partial_j := \frac{\partial}{\partial x_j}$.

Let $d_t := \partial_t + \mathbf{u} \cdot \nabla$ denote the **convective derivative**.

Let $\eth_t := \alpha \mapsto (\partial_t \alpha + \nabla \cdot (\mathbf{u}\alpha))$ denote the **conservative derivative**. (I made up this term and this symbol. $\eth_t$ is supposed to be reminiscent of the averaging operator $\fint$ and δ signifying differentiation.)

Leibnitz rules.

Observe that $\eth_t \alpha = d_t \alpha + (\nabla \cdot \mathbf{u})\alpha$. Hence:

$d_t(\alpha\beta) = (d_t\alpha)\beta + \alpha(d_t\beta)$.

$\eth_t(\alpha\beta) = d_t(\alpha\beta) + (\nabla \cdot \mathbf{u})\alpha\beta$
$= (d_t\alpha)\beta + \alpha(d_t\beta) + (\nabla \cdot \mathbf{u})\alpha\beta$
$= (\eth_t\alpha)\beta + \alpha(d_t\beta)$
$= (d_t\alpha)\beta + \alpha(\eth_t\beta)$.

Gauss's Theorem

$\int \nabla\alpha = \oint \mathbf{n}\alpha$, $\boxed{\int \nabla \cdot \mathbf{q} = \oint \mathbf{n} \cdot \mathbf{q}}$, and $\int \nabla \times \mathbf{q} = \oint \mathbf{n} \times \mathbf{q}$.

Reynolds' Transport Theorem.

$\boxed{\frac{d}{dt} \int \alpha = \int \eth_t \alpha}$, i.e.

$\frac{d}{dt} \int_{U(t)} \alpha = \int_{U(t)} (\partial_t \alpha + \nabla \cdot (\mathbf{u}\alpha))$

Justification. (Convection applies to $U(t)$, not $\alpha(\mathbf{x}, t)$.) Use time-splitting on the time increment: alternatively allow α and $U(t)$ to evolve. Then apply Gauss's Theorem.

$\frac{d}{dt} \int_{U(t)} \alpha = \int \partial_t \alpha + \oint \mathbf{n} \cdot \mathbf{u}\alpha = \int \partial_t \alpha + \int \nabla \cdot (\mathbf{u}\alpha))$

1.3 Conservation Laws

1.3.1 Definitions of Quantities

Let ρ denote mass per volume.

Observe that $\mathbf{u}$ is momentum per mass.

Let e denote internal (i.e. thermal) energy per mass.

Observe that $\frac{1}{2}u^2$ is macroscopic kinetic energy per mass.

Let $\mathbf{g}$ denote body force (force per unit mass).

Let $\underline{\underline{\tau}}$ denote the stress tensor: $\mathbf{n} \cdot \underline{\underline{\tau}}$ is the surface force per unit area on an infinitesimal surface element orthogonal to $\mathbf{n}$, where $\mathbf{n}$ points away from the side of the interface on which the force acts. Thus $\tau_{ij} := \mathbf{e}_i \cdot \underline{\underline{\tau}} \cdot \mathbf{e}_j$ is the component in the direction $\mathbf{e}_j$ of the surface force acting on the low side of an infinitesimal surface orthogonal to $\mathbf{e}_i$. This stress tensor representation of surface forces is justified by noting that the sum of the forces must be zero on an infinitesimal tetrahedron with 3 sides aligned with the principle axes. Application of conservation of angular momentum to an infinitesimal cube aligned with the principle axes shows that the stress tensor is symmetric. (See Aris or Borisenko and Tarapov.)

Let $\mathbf{q}$ denote the heat flux: $\mathbf{q} \cdot \mathbf{n}$ is the rate of external flow of heat per unit area across an infinitesimal surface element orthogonal to $\mathbf{n}$ (i.e. the component of the flow of heat in the direction of $\mathbf{n}$).

1.3.2 Conservation of Mass

$\frac{d}{dt} \int \rho = 0$, i.e. $\boxed{\eth_t \rho = 0}$, i.e. $\boxed{\partial_t \rho + \nabla \cdot \rho\mathbf{u} = 0}$.

1.3.3 Balance of Momentum

$\frac{d}{dt} \int \rho\mathbf{u} = \oint \mathbf{n} \cdot \underline{\underline{\tau}} + \int \rho\mathbf{g}$, i.e.

$\boxed{\eth_t(\rho\mathbf{u}) = \nabla \cdot \underline{\underline{\tau}} + \rho\mathbf{g}}$ (conservation form).

Simplify using Leibnitz rule and conservation of mass: $\eth_t(\rho\mathbf{u}) = \cancel{(\eth_t\rho)}\mathbf{u} + \rho(d_t\mathbf{u})$. So:

$\boxed{\rho d_t \mathbf{u} = \nabla \cdot \underline{\underline{\tau}} + \rho\mathbf{g}}$ (simplified form).

1.3.4 Balance of Kinetic Energy

Dot momentum balance with $\mathbf{u}$.

Use $\rho(d_t\mathbf{u}) \cdot \mathbf{u} = \rho d_t(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) = \eth_t(\frac{1}{2}\rho u^2)$.

Get: $\eth_t(\frac{1}{2}\rho u^2) = (\nabla \cdot \underline{\underline{\tau}}) \cdot \mathbf{u} + \rho\mathbf{g} \cdot \mathbf{u}$.

1.3.5 Conservation of Energy

$\frac{d}{dt} \int (\rho e + \frac{1}{2}\rho u^2) = \oint \mathbf{n} \cdot \underline{\underline{\tau}} \cdot \mathbf{u} + \int \rho\mathbf{g} \cdot \mathbf{u} - \oint \mathbf{n} \cdot \mathbf{q}$, i.e.

$\boxed{\eth_t(\rho e + \frac{1}{2}\rho u^2) = \nabla \cdot (\underline{\underline{\tau}} \cdot \mathbf{u}) + \rho\mathbf{g} \cdot \mathbf{u} - \nabla \cdot \mathbf{q}}$

(energy balance)

We decouple thermal from kinetic energy conservation by subtracting the kinetic energy balance equation.

We use $\nabla \cdot (\underline{\underline{\tau}} \cdot \mathbf{u}) = \frac{\partial}{\partial x_i}(\tau_{ij}\mathbf{u}_j) = \tau_{ij}\frac{\partial}{\partial x_i}\mathbf{u}_j + (\frac{\partial}{\partial x_i}\tau_{ij})\mathbf{u}_j$
$= \underline{\underline{\tau}} : \nabla\mathbf{u} + (\nabla \cdot \underline{\underline{\tau}}) \cdot \mathbf{u}$
(":" here denotes contraction of two indices).

So $\boxed{\rho d_t e = \underline{\underline{\tau}} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}}$ (heat balance)

To show energy conservation, assume a potential: $\mathbf{g} = -\nabla\chi$. Using $\rho\mathbf{u} \cdot \nabla\chi = \nabla \cdot (\rho\mathbf{u}\chi) - \nabla \cdot (\rho\mathbf{u})\chi = \nabla \cdot (\rho\mathbf{u}\chi) + (\partial_t\rho)\chi = \nabla \cdot (\mathbf{u}(\rho\chi) + \partial_t(\rho\chi) - \rho\partial_t\chi$ gives:

$\boxed{\eth_t(\rho e + \frac{1}{2}\rho u^2 + \rho\chi) = \nabla \cdot (\underline{\underline{\tau}} \cdot \mathbf{u} - \mathbf{q}) + (\partial_t\chi)\rho}$

(energy conservation)

2 Constitutive Relations

2.1 Isotropic linear fluid

Assume that $\underline{\underline{\tau}} = -p\underline{\underline{\delta}} + \underline{\underline{\sigma}}$, where

p = pressure,
$\underline{\underline{\delta}}$ = identity tensor (2nd order), and
$\underline{\underline{\sigma}}$ = viscous/shear stress tensor

The viscous stress tensor is assumed to depend linearly on the velocity gradient tensor $\nabla\mathbf{u}$.

So $\sigma_{ij} = K_{ijkl}\partial_k u_l$ for some fourth-order tensor K. Assume that K is isotropic. Then K_{ijkl} is a linear combination of products of δs[1]:

$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$.

So $\sigma_{ij} = \lambda\delta_{ij}\nabla\cdot\mathbf{u} + \mu\nabla\mathbf{u} + \nu(\nabla\cdot\mathbf{u})^T$. But since σ is symmetric, we must have $\mu = \nu$,

so $\boxed{\underline{\underline{\sigma}} = (\lambda\nabla\cdot\mathbf{u})\underline{\underline{\delta}} + 2\mu\,\mathrm{Sym}(\nabla\mathbf{u})}$, where $\mathrm{Sym}(\nabla\mathbf{u}) := (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ is the *even part* of the velocity gradient, called the **rate-of-strain** or **rate of deformation** tensor. In other words, the symmetry of the stress tensor means that it depends only on the *symmetric part* of the velocity gradient.[2]

The Stokes assumption asserts that the trace of the viscous stress tensor is zero.[3] So the scalar pressure p is (defined to be) minus one third the trace of the stress tensor $\underline{\underline{\tau}}$. This ensures that the translational thermal energy is $3/2$ the pressure.

So $\boxed{\underline{\underline{\tau}} = -p\underline{\underline{\delta}} + 2\mu\left(\mathrm{Sym}(\nabla\mathbf{u}) - \nabla\cdot\mathbf{u}\,\underline{\underline{\delta}}/3\right)}$.

2.2 Incompressible Navier-Stokes.

Assume that the fluid is incompressible: $\boxed{\nabla\cdot\mathbf{u} = 0.}$

Then $\nabla\cdot\underline{\underline{\sigma}} = \mu\triangle\mathbf{u}$.

So $\boxed{\rho d_t\mathbf{u} = \mu\triangle\mathbf{u} - \nabla p + \rho\mathbf{g}.}$

2.3 Euler equations.

"Eulerian flow" generally refers to flow where there are no dissipative processes (i.e. no second derivatives). So Eulerian flow assumes that the viscous stress tensor is zero; equivalently $\mu = 0$ and $\lambda = 0$. Hence:

$\boxed{\rho d_t\mathbf{u} = -\nabla p + \rho\mathbf{g}.}$

[1] Sir Harold Jeffreys. *On isotropic tensors.* Proc. Camb. Phil. Soc. (1973), **73**, 173.

[2] There is a physical reason that the stress tensor depends only on the symmetric part. Constant antisymmetric velocity gradient tensors correspond bijectively with rigid-body rotations. The viscous stress tensor is zero for a fluid undergoing rigid-body rotation. Since the viscous stress is a linear function of the velocity gradient, it is the sum of this function evaluated on its symmetric and antisymmetric parts, so it is a linear function merely of its symmetric part.

[3] The Stokes assumption is equivalent to the assumption that intraspecies collisions exchange no energy with non-translational modes. I discovered this by comparing the 10-moment isotropic intraspecies collisional closure for thermal energy with the 5-moment isotropic viscous stress closure, as detailed in my note on the ten-moment closure.

2.3.1 Isentropic Euler.

If the flow is compressible, we can close the system by specifying a relationship between p and ρ. Pressure is indeed a function of density in the case of the adiabatic assumption that there is no heat flow, i.e. entropy is conserved. For an ideal gas, this implies $\boxed{p = C\rho^\gamma,}$ where γ is the ratio of specific heats (i.e. adiabatic/polytropic index) and C remains constant along particle paths.

2.3.2 Conservative Euler.

For flow in which shock waves occur, entropy is not conserved, so the adiabatic assumption breaks down and we must replace the conservation of entropy with the conservation of energy.

For an ideal gas, the internal energy per volume is $\rho e = \frac{p}{\gamma-1}$. Let E denote the energy per volume. So $\boxed{E = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2.}$ This supplies a relationship between p and ρ at the cost of introducing a new variable E. But since E is conserved, we can get a closed system if we specify the heat flux $\mathbf{q} = \kappa\nabla\theta^{-1}$, where $\theta = p/\rho$ is proportional to temperature and κ is equivalent to heat conductivity.

3 Bibliography

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