Special relativity derived from classical electrodynamics

by Alec Johnson, February 2011

This document derives relativistic electrodynamics from classical electrodynamics and the invariance of the interval defined by the Lorentz metric. See my note on the special relativistic metric for background.

1 Background: classical electrodynamics

We begin by reciting the equations of classical electrodynamics. For a derivation of these laws see my derivation of the laws of classical physics titled, *What is vector calculus good for?*

Classical electrodynamics is governed by **Newton's second law** (for particle motion),

$$m_{\mathrm{p}} d_t \mathbf{v}_{\mathrm{p}} = \mathbf{F}_{\mathrm{p}}, \qquad \mathbf{v}_{\mathrm{p}} := d_t \mathbf{x}_{\mathrm{p}},$$

(where p is particle index, t is time, m_{p} is particle mass, $\mathbf{x}_{\mathrm{p}}(t)$ is particle position, and $\mathbf{F}_{\mathrm{p}}(t)$ is the force on the particle), **the Lorentz force law**,

$$\mathbf{F}_{\mathrm{p}} = q_{\mathrm{p}} \left(\mathbf{E} + \mathbf{v}_{\mathrm{p}} \times \mathbf{B} \right)$$

(where $\mathbf{E}(t, \mathbf{x})$ = electric field, $\mathbf{B}(t, \mathbf{x})$ = magnetic field, and q_{p} is particle charge), and **Maxwell's equations**,

$$\begin{aligned} &\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, && \nabla \cdot \mathbf{B} = 0, \\ &\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0, && \nabla \cdot \mathbf{E} = \sigma/\epsilon_0, \\ &\mathbf{J} := \sum_{\mathrm{p}} q_{\mathrm{p}} \mathbf{v}_{\mathrm{p}} \delta_{\mathbf{x}_{\mathrm{p}}}, && \sigma := \sum_{\mathrm{p}} q_{\mathrm{p}} \delta_{\mathbf{x}_{\mathrm{p}}}, \end{aligned}$$

where c is light speed, ϵ_0 is electric permittivity, $\mathbf{J}$ is net current density, σ is net charge density, $\delta_{\mathbf{x}_{\mathrm{p}}}(t, \mathbf{x}) := \delta(\mathbf{x} - \mathbf{x}_{\mathrm{p}}(t))$ is the particle density function, and δ is the Dirac delta function (unit spike).

2 Overview

The purpose of this document is to show that the equations of classical electrodynamics given in the opening section become Lorentz-invariant (i.e. invariant under inertial transformations) merely by replacing the velocity $\mathbf{v}_{\mathrm{p}}$ with proper velocity $\gamma_{\mathrm{p}} \mathbf{v}_{\mathrm{p}}$ in Newton's second law, where $\gamma_{\mathrm{p}} := 1/\sqrt{1 - (\mathbf{v}_{\mathrm{p}}/c)^2}$ is the Lorentz factor. (In other words, the equations of special relativity are the equations of classical electrodynamics with particle momentum $m_{\mathrm{p}} \mathbf{v}_{\mathrm{p}}$ redefined to be $m_{\mathrm{p}} \gamma_{\mathrm{p}} \mathbf{v}_{\mathrm{p}}$.) With this one modification we will show how to put the fundamental equations of electrodynamics in a *covariant* (i.e. manifestly Lorentz-invariant) form.

Newton's second law is invariant under Galilean transformations, but Maxwell's equations are not (because they imply the existence of a light speed c).

Einstein (and previously Lorentz) characterized the set of affine coordinate transformations, called Lorentz transformations (or inertial transformations), which leave light speed constant and which satisfy the property that if system A has velocity $\mathbf{v}$ in system B then system B has velocity $-\mathbf{v}$ in system A.

As I argue in my note on the special relativistic metric, the set of Lorentz transformations is characterized by the property that they preserve the **interval** between events:

$$(c\, \mathrm{d}t)^2 - \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}x^\mu \mathrm{d}x_\mu,$$

where, in accordance with the Einstein summation convention, there is an implicit sum over the index μ because it appears twice in its term both as a superscript and as a subscript. (We adopt the convention that Greek indices run from 0 to 3 and Latin indices run from 1 to 3.) The four-vector $\mathrm{d}x^\mu$ has components $(dx^0 = c\mathrm{d}t, dx^1, dx^2, dx^3)$ and its dual four-position has components $(x_0 = x^0, x_1 = -x^1, x_2 = -x^2, x_3 = -x^3)$. All of special relativity flows from the invariance of this interval.

Classical electrodynamics is not relativistic. In particular, Maxwell's equations are not Galilean-invariant (because they imply the existence of a light speed; in fact, we will see that they are already Lorentz-invariant), whereas Newton's equations are Galilean-invariant (and therefore not Lorentz-invariant).

3 Newton's second law

We first modify Newton's equations to make them Lorentz-invariant. To do so we replace vectors with four-vectors and replace d_t, the derivative with respect to coordinate time t, with d_τ, the derivative with respect to *proper time* τ, defined to be the rate of elapse of time in the reference frame of the particle.

Then Newton's second law becomes

$$m_{\mathrm{p}} \mathrm{d}_\tau \widetilde{\mathbf{v}}_{\mathrm{p}}^\mu = \widetilde{\mathbf{F}}_{\mathrm{p}}^\mu, \qquad \widetilde{\mathbf{v}}_{\mathrm{p}}^\mu := \mathrm{d}_\tau \mathbf{x}_{\mathrm{p}}^\mu,$$

where $\widetilde{\mathbf{v}}$ is called the *proper velocity* and $\widetilde{\mathbf{F}}$ is called the *Minkowski force*. (We will see that the index value $\mu = 0$ makes sense here.) Proper time is by definition independent of reference frame, so this version of Newton's law is covariant.

3.1 Lorentz factor

We can use the chain rule to relate proper derivatives to time derivates:

$$\mathrm{d}_\tau = \gamma \mathrm{d}_t,$$

where $\gamma := \frac{\mathrm{d}t}{\mathrm{d}\tau}$ is called the *Lorentz factor*. Thus

$$\widetilde{\mathbf{v}} = \gamma \mathbf{v}.$$

To obtain a formula for γ we begin with the invariance of the metric, which implies that an infinitesimal interval in the frame of the particle equals the interval in observer coordinates:

$$(c\,\mathrm{d}\tau)^2 = (c\,\mathrm{d}t)^2 - (\mathrm{d}\mathbf{x})^2, \text{ i.e.,}$$
$$(c\,\mathrm{d}t)^2 = (c\,\mathrm{d}\tau)^2 + (\mathrm{d}\mathbf{x})^2$$

Dividing the second equation by $(c\,\mathrm{d}\tau)^2$ relates the Lorentz factor to proper velocity:

$$\gamma^2 = 1 + (\widetilde{\mathbf{v}}/c)^2.$$

Differentiating gives a very useful differential relation,

$$\gamma \mathrm{d}\gamma = \frac{\widetilde{\mathbf{v}}}{c} \cdot \mathrm{d}\frac{\widetilde{\mathbf{v}}}{c}, \text{i.e.,}$$
$$\mathrm{d}\gamma = \frac{\mathbf{v}}{c} \cdot \mathrm{d}\frac{\widetilde{\mathbf{v}}}{c}.$$

Dividing the first equation by $(c\,\mathrm{d}t)^2$ relates the Lorentz factor to ordinary velocity:

$$\gamma^{-2} = 1 - (\mathbf{v}/c)^2.$$

3.2 Acceleration

We call $\widetilde{\mathbf{a}} := \mathrm{d}_\tau \widetilde{\mathbf{v}}_\mathrm{p}$ the **proper acceleration**; the (relativistic) **acceleration** is defined to be $\mathbf{a}_\mathrm{p} := \mathrm{d}_t \widetilde{\mathbf{v}}_\mathrm{p}$.

We can infer the zeroth component of the acceleration from the spatial components of acceleration. Indeed,

$$\begin{aligned}
x^0 &= ct, \\
\widetilde{\mathbf{v}}^0 &= \gamma \mathrm{d}_t x^0 = \gamma c, \\
\mathbf{a}^0 &= \mathrm{d}_t \widetilde{\mathbf{v}}^0 = c\mathrm{d}_t \gamma = \mathbf{v} \cdot \mathrm{d}_t \widetilde{\mathbf{v}}/c \\
&= \mathbf{v} \cdot \mathbf{a}/c, \\
\widetilde{\mathbf{a}}^0 &= \gamma \mathbf{a}^0 \\
&= \mathbf{v} \cdot \widetilde{\mathbf{a}}/c,
\end{aligned}$$

4 Electromagnetic potential evolution.

We will need to determine how to modify the formula for the Lorentz force to make it covariant. First we need to express electromagnetic field in covariant fashion. The easiest motivated way I know how to do this is to formulate Maxwell's equations in terms of vector potentials. (The potential satisfies a wave equation with velocity c, which we will see makes it Lorentz-invariant, whereas Maxwell's equations as written above are not manifestly Lorentz-invariant — and in fact under Lorentz transformation the components of the electric and magnetic field do not transform like vectors.)

To define potentials we use the homogeneous Maxwell equations. $\nabla \cdot \mathbf{B} = 0$ implies that we can write $\mathbf{B} = \nabla \times \mathbf{A}$ for a vector potential $\mathbf{A}$. Substituting into Faraday's law, $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$, gives $\nabla \times (\partial_t \mathbf{A} + \mathbf{E}) = 0$, which says that we can write $\partial_t \mathbf{A} + \mathbf{E} = -\nabla \phi$.

To obtain evolution equations for the potentials $\mathbf{A}$ and ϕ we make the substitutions $\mathbf{B} \to \nabla \times \mathbf{A}$ and $\mathbf{E} \to -\partial_t \mathbf{A} - \nabla \phi$ in the nonhomogenous Maxwell equations. Electric field evolution $-\frac{\partial \mathbf{E}}{\partial t} + c^2 \nabla \times \mathbf{B} = \mathbf{J}/\epsilon_0$ becomes

$$\partial_t^2 \mathbf{A} - c^2 \nabla^2 \mathbf{A} + \nabla \left(\partial_t \phi + c^2 \nabla \cdot \mathbf{A} \right) = \mathbf{J}/\epsilon_0 \tag{1}$$

and the electric field divergence constraint $\nabla \cdot \mathbf{E} = \sigma/\epsilon_0$ becomes

$$-\nabla^2 \phi - \nabla \cdot \partial_t \mathbf{A} = \sigma/\epsilon_0.$$

To see that this is Lorentz-invariant, we divide equation (1) by c^2. Using that

$$\partial_{(ct)}(\phi/c) + \nabla \cdot \mathbf{A} = \partial_\mu A^\mu,$$

where the four potential is defined by

$$A^\mu := (\phi/c, \mathbf{A})^\mathrm{T},$$

and adopting the shorthand

$$\partial_\mu := \partial_{\mathbf{x}^\mu} = (\partial_t, \nabla)^\mathrm{T} \quad \text{and} \quad \partial^\mu = \partial_{\mathbf{x}_\mu} = (\partial_t, -\nabla)^\mathrm{T},$$

so that the D'Alembertian may be expressed by

$$\partial_\mu \partial^\mu = \partial_{(ct)}{}^2 - \nabla^2,$$

electric field evolution says

$$\boxed{\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = \mu_0 J^\nu}, \tag{2}$$

where we use the Greek index ν for the free index to anticipate that this holds even for $\nu = 0$. Indeed, in case $\nu = 0$ equation (2) reduces to

$$\partial_j \partial^j A^0 - \partial^0 \partial_j A^j = \mu_0 J^0, \tag{3}$$

which is the electric field divergence constraint if we define the four-current by

$$J^\mu = (c\sigma, \mathbf{J})^\mathrm{T}.$$

If we impose the (Lorentz-invariant) Lorentz gauge $\partial_\mu A^\mu = 0$ then electric field evolution simplifies to

$$\boxed{\partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu}.$$

For justification that one can impose the Lorentz gauge see the appendix.

5 Force law

We need to put the Lorentz force law into covariant form. Since we have a covariant representation of potentials, we can do so by substituting the potential representations $\mathbf{B} \to$

$\nabla\times\mathbf{A}$ and $\mathbf{E}\to-\partial_t\mathbf{A}-\nabla\phi$. As a shortcut we generalize the Lorentz force law in case $\mathbf{E}=0$:

$$\widetilde{\mathbf{F}}/q=\widetilde{\mathbf{v}}\times\mathbf{B}.$$

Substituting $\mathbf{B}=\nabla\times\mathbf{A}$,

$$\widetilde{\mathbf{F}}^j/q=\widetilde{\mathbf{v}}^k(\partial^j A_k-\partial_k A^j)$$

The natural generalization is to replace Latin indices with Greek indices. So defining the antisymmetric electromagnetic field tensor

$$\mathcal{F}^\mu{}_\nu:=\partial^\mu A_\nu-\partial_\nu A^\mu$$

we conjecture the Lorentz force law

$$\boxed{\widetilde{\mathbf{F}}^\mu/q=\widetilde{\mathbf{v}}^\nu\mathcal{F}^\mu{}_\nu}.$$

Indeed, for spatial indices $\mu=j$ this is

$$\begin{aligned}\widetilde{\mathbf{F}}^j/q&=\widetilde{\mathbf{v}}^\nu(\partial^j A_\nu-\partial_\nu A^j)\\&=(\gamma c)(\partial^j A_0-\partial_0 A^j)+\widetilde{\mathbf{v}}^k(\partial^j A_k-\partial_k A^j)\\&=-(\gamma c)(\nabla(\phi/c)+\partial_{(ct)}\mathbf{A})+\widetilde{\mathbf{v}}\times(\nabla\times\mathbf{A})\\&=\gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B}),\end{aligned}$$

which agrees exactly with the original (classical) Lorentz force law, and for $\mu=0$ it says

$$\begin{aligned}\widetilde{\mathbf{F}}^0/q&=\widetilde{\mathbf{v}}^\nu(\partial^0 A_\nu-\partial_\nu A^0)\\&=\widetilde{\mathbf{v}}^k(\partial^0 A_k-\partial_k A^0)\\&=\gamma\mathbf{v}\cdot(-\partial_{(ct)}\mathbf{A}-\nabla(\phi/c))\\&=\gamma\mathbf{E}\cdot\mathbf{v}/c\\&=m\widetilde{\mathbf{a}}^0/q,\end{aligned}$$

as needed.

For reference, the components of the electromagnetic field tensor are:

$$\mathcal{F}^\mu{}_\nu=\begin{bmatrix}0&E^1/c&E^2/c&E^3/c\\E^1/c&0&B^3&-B^2\\E^2/c&-B^3&0&B^1\\E^3/c&B^2&-B^1&0\end{bmatrix}=\begin{bmatrix}0&\mathbf{E}^T/c\\\mathbf{E}/c&-\mathbf{B}\times\mathbb{I}\end{bmatrix};$$

or, in manifestly antisymmetric components,

$$c\mathcal{F}^{\mu\nu}=\begin{bmatrix}0&-E^1&-E^2&-E^3\\E^1&0&-cB^3&cB^2\\E^2&cB^3&0&-cB^1\\E^3&-cB^2&cB^1&0\end{bmatrix}=\begin{bmatrix}0&-\mathbf{E}^T\\\mathbf{E}&c\mathbf{B}\times\mathbb{I}\end{bmatrix}.$$

6 Definition of current

It remains to verify that the definition of current and charge density (i.e. four-current) remains unmodified when put in covariant form. We rewrite current and charge density as

$$\begin{aligned}\mathbf{J}&=\sum_{\mathrm{p}}q_{\mathrm{p}}\mathbf{v}_{\mathrm{p}}\delta_{\mathbf{x}_{\mathrm{p}}} & \sigma&=\sum_{\mathrm{p}}q_{\mathrm{p}}\delta_{\mathbf{x}_{\mathrm{p}}}\\&=\sum_{\mathrm{p}}q_{\mathrm{p}}\widetilde{\mathbf{v}}_{\mathrm{p}}\frac{\delta_{\mathbf{x}_{\mathrm{p}}}}{\gamma_{\mathrm{p}}}, & &=\sum_{\mathrm{p}}q_{\mathrm{p}}\gamma_{\mathrm{p}}\frac{\delta_{\mathbf{x}_{\mathrm{p}}}}{\gamma_{\mathrm{p}}}.\end{aligned}$$

We need that $\frac{\delta_{\mathbf{x}_{\mathrm{p}}}}{\gamma_{\mathrm{p}}}$ is Lorentz-invariant. Recall that delta functions are defined by their integrals. Since $\int_\Omega\left(\frac{\delta_{\mathbf{x}_{\mathrm{p}}}}{\gamma_{\mathrm{p}}}\right)(\gamma_p d^3\mathbf{x})=1$, we need that to show that $\gamma_p d^3\mathbf{x}$ is Lorentz-invariant. Recall that any Lorentz transformation can be represented as a Lorentz boost along the $\mathbf{x}$ axis preceeded and followed by a rotation. So it is enough to consider a boost along the x axis with Lorentz factor γ_p which takes an infinitesimal region of width $\mathrm{d}x$ of simultaneous points ($\mathrm{d}t=0$) in space time and applies a boost with a Lorentz factor of γ_p. The transformed region has width $\mathrm{d}x'=\gamma_p\mathrm{d}x$. I remark that the transformed region is not simultaneous (in fact, $\mathrm{d}t'=\mathrm{d}x\sqrt{1-\gamma_{\mathrm{p}}^2}$).

To make this argument concrete, consider a stream of particles with identical velocity $\widetilde{\mathbf{v}}$. Take the primed frame of reference to be the "rest frame" (the frame in which the particles are at rest). Let N be the number of particles crossing the event region $(\mathrm{d}t,\mathrm{d}x)$ in the "laboratory frame". N is unchanged under change of reference frame. Projecting $(\mathrm{d}t',\mathrm{d}x')$ onto a simultaneous region (i.e. resetting $\mathrm{d}t'=0$) also does not change the number of particles crossing the event region (because their velocity is approximately zero). Let n_0 be the density of particles in the primed reference frame. Then the number of particles N intersecting the event region is $N=n_0\mathrm{d}x'=n_0(\gamma\mathrm{d}x)$. Both N and n_0 are canonically defined and thus Lorentz-invariant, so $\gamma\mathrm{d}x$ must be as well. I remark that we can also conclude that particle density in the laboratory frame is $n=\gamma n_0$, which is relevant if you consider it more natural to *define* current as flux of charged particles and wish to show that this definition is Lorentz-invariant. Taking this viewpoint, we note that the current density in the laboratory rest frame is $\mathbf{J}=q_{\mathrm{p}}n\mathbf{v}_{\mathrm{p}}=q_{\mathrm{p}}n_0\widetilde{\mathbf{v}}_{\mathrm{p}}$. The covariant generalization $J^\mu=q_{\mathrm{p}}n_0\widetilde{\mathbf{v}}_{\mathrm{p}}^\mu$ thus holds if it holds for $\mu=0$, i.e. if $c\sigma=q_{\mathrm{p}}n_0\gamma c$, which just says that $\sigma=q_{\mathrm{p}}n$.

A Lorentz gauge condition

There is gauge freedom in the choice of potentials. Indeed, suppose that A' and ϕ' satisfy $\mathbf{B}=\nabla\times\mathbf{A}'$ and $\partial_t\mathbf{A}'+\mathbf{E}=-\nabla\phi'$ and so do $\mathbf{A}$ and ϕ. Then

$$A=A'+\nabla\lambda\qquad\text{and}\qquad\phi=\phi'-\partial_t\lambda$$

(using that we can absorb a constant of integration into λ).

Lorentz gauge. Assume that $\mathbf{A}'$ and ϕ' are arbitrary potentials (not satisfying any particular gauge condition). We want to choose λ so that the Lorentz gauge condition holds, $\partial_t\phi+c^2\nabla\cdot\mathbf{A}=0$. Substituting, $\partial_t\phi'-\partial_t{}^2\lambda+c^2\nabla\cdot\mathbf{A}'+c^2\nabla^2\lambda=0$, i.e.,

$$\partial_t{}^2\lambda-c^2\nabla^2\lambda=\partial_t\phi'+c^2\nabla\cdot\mathbf{A}'.$$

This is a wave equation which we can solve for λ (for arbitrary choice of λ_0 and $(\partial_t\lambda)_0$, so properly we should speak of *a* Lorentz gauge rather than *the* Lorentz gauge).