The Relativistic Vlasov Equation

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I. Recitation of basic electrodynamics

A. Classical electrodynamics

Classical electrodynamics is governed by **Newton's second law** (for particle motion),

$$m_p d_t \mathbf{v}_p = \mathbf{F}_p, \qquad \mathbf{v}_p := d_t \mathbf{x}_p,$$

(where p is particle index, t is time, m_p is particle mass, $\mathbf{x}_p(t)$ is particle position, and $\mathbf{F}_p(t)$ is the force on the particle), **the Lorentz force law**,

$$\mathbf{F}_p = q_p (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(where $\mathbf{E}(t, \mathbf{x})$ = electric field, $\mathbf{B}(t, \mathbf{x})$ = magnetic field, and q_p is particle charge), and **Maxwell's equations**,

$$\begin{aligned} &\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, && \nabla \cdot \mathbf{B} = 0, \\ &\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon, && \nabla \cdot \mathbf{E} = \sigma/\epsilon, \end{aligned}$$

with current and charge density given by

$$\mathbf{J} := \sum_p q_p \mathbf{v}_p \delta_{\mathbf{x}_p}, \qquad \sigma := \sum_p q_p \delta_{\mathbf{x}_p},$$

where c is light speed, ϵ_0 is electric permittivity, $\mathbf{J}$ is net current density, σ is net charge density, $\delta_{\mathbf{x}_p}(t, \mathbf{x}) := \delta(\mathbf{x} - \mathbf{x}_p(t))$ is the particle density function, and δ is the Dirac delta function (unit spike).

B. Relativistic electrodynamics

Newton's second law is invariant under Galilean transformations, but Maxwell's equations are not (because they imply the existence of a light speed c).

Einstein (and previously Lorentz) characterized the set of affine coordinate transformations, called Lorentz transformations, which leave light speed constant and which satisfy the property that if system A has velocity $\mathbf{v}$ in system B then system B has velocity $-\mathbf{v}$ in system A.

Einstein then modified the laws of electrodynamics to make them Lorentz-invariant. To do so we replace velocity with proper velocity in Newton's second law. So the laws of relativistic electrodynamics are:

Newton's second law,

$$m_p d_t \mathbf{p}_p = \mathbf{F}_p, \qquad \mathbf{v}_p := d_t \mathbf{x}_p,$$

the **Lorentz force law**,

$$\mathbf{F}_p = q_p (\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

and **Maxwell's equations**,

$$\begin{aligned} &\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, && \nabla \cdot \mathbf{B} = 0, \\ &\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon, && \nabla \cdot \mathbf{E} = \sigma/\epsilon, \end{aligned}$$

with current and charge density given by

$$\begin{aligned} \mathbf{J} &= \sum_p q_p \mathbf{v}_p \delta_{\mathbf{x}_p} & \sigma &= \sum_p q_p \delta_{\mathbf{x}_p} \\ &= \sum_p q_p \mathbf{p}_p \frac{\delta_{\mathbf{x}_p}}{\gamma_p}, & &= \sum_p q_p \gamma_p \frac{\delta_{\mathbf{x}_p}}{\gamma_p}; \end{aligned}$$

here $\mathbf{p}_p$ (i.e. $\gamma_p \mathbf{v}_p$) is the proper velocity, where γ_p is the rate at which time t elapses with respect to the proper time τ_p of a clock moving with particle p. Dropping the particle index,

$$\begin{aligned} &\gamma := d_\tau t, \quad \text{and} \\ &\mathbf{p} := d_\tau \mathbf{x} = (d_\tau t) d_t \mathbf{x} = \gamma \mathbf{v}. \end{aligned}$$

A concrete expression relating γ to (proper) velocity is

$$\begin{aligned} &\gamma^2 = 1 + (\mathbf{p}/c)^2, \quad \text{i.e. (dividing by } \gamma^2), \\ &1 = \gamma^{-2} + (v/c)^2, \quad \text{i.e. (solving for } \gamma), \\ &\gamma = \left(1 - (v/c)^2\right)^{-1/2}. \end{aligned}$$

II. Vlasov equation

The Vlasov equation asserts that particles are conserved in phase space and that the only force acting on particles is the electromagnetic force.

The Vlasov equation writes

$$\partial_t f_s + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{p}} \cdot (\mathbf{a}_s f_s) = 0, \tag{1}$$

where $f_s(t, \mathbf{x}, \mathbf{p})$ is the particle density function of species s and $\mathbf{a}_s = d_t \mathbf{p} = (q_s/m_s)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is particle acceleration and we recall that $\mathbf{v} = d_t \mathbf{x}$ is velocity and $\mathbf{p} = \gamma \mathbf{v}$ is proper velocity.

The Maxwell source terms are provided by the relations

$$\begin{aligned} \mathbf{J} &= \sum_s q_s \int f_s \mathbf{v} d\mathbf{p} & \sigma &:= \sum_s q_s \int f_s d\mathbf{p} \\ &= \sum_s q_s \int f_s \mathbf{p} \frac{d\mathbf{p}}{\gamma}, & &= \sum_s q_s \int f_s \gamma \frac{d\mathbf{p}}{\gamma}. \end{aligned}$$

If we regard f_s as a linear superposition of spike functions then the Vlasov system (1) with Maxwell's equations is equivalent to the fundamental equations of electrodynamics written in terms of individual particles.

A. Lorentz-invariant form

1. f_s is a scalar.

(In this section without loss of generality we take $c = 1$.) The particle density function $f_s(t, \mathbf{x}, \mathbf{p})$ is a scalar. This means that it remains invariant under the transformation of phase space that is implied by a Lorentz transformation (i.e. a relativistic inertial transformation). It takes a little work to see this. By definition, the quantity $\mathrm{d}f_\mathrm{s} := f_\mathrm{s}(t,\mathbf{x},\mathbf{p})\mathrm{d}^3\mathbf{x}\wedge\mathrm{d}^3\mathbf{p}$ counts the number of particles in a region $(\mathbf{x}_0+\mathrm{d}^3\mathbf{x},\mathbf{p}_0+\mathrm{d}^3\mathbf{p})$ in phase space. (Note that $\mathrm{d}^3\mathbf{x}\wedge\mathrm{d}^3\mathbf{p}$ denotes an infinitesimal surface of simultaneous points in $(t,\mathbf{x},\mathbf{p})$-space.) We argue that $\mathrm{d}f_\mathrm{s}$ is a scalar, then argue that $\mathrm{d}^3\mathbf{x}\wedge\mathrm{d}^3\mathbf{p}$ is a scalar, and then conclude that f_s is a scalar. The essence of the argument is to consider a Lorentz boost which transforms into the reference frame of $\mathrm{d}^3\mathbf{x}\wedge\mathrm{d}^3\mathbf{p}$ (i.e. which transforms $\mathbf{p}_0+\mathrm{d}^3\mathbf{p}$ to $\mathrm{d}^3\mathbf{p}'$, so that the transformed range of velocities is centered on zero).

To see that such a boost leaves $\mathrm{d}f_\mathrm{s}$ invariant, consider the particle event paths ("world lines") that intersect $\mathrm{d}^3\mathbf{x}\wedge\mathrm{d}^3\mathbf{p}$. The transformed region of phase space will be intersected by exactly the same particles. Although the transformed region of phase space is not simultaneous, orthogonal projection to make it simultaneous does not change the number of particles intersecting it because their velocity is approximately zero.

To see that the same boost leaves $\mathrm{d}^3\mathbf{x}\wedge\mathrm{d}^3\mathbf{p}$ invariant, note that orthogonality of $\mathrm{d}^3\mathbf{x}$ and $\mathrm{d}^3\mathbf{p}$ is invariant under the boost, that the simultaneous region $\mathrm{d}^3\mathbf{x}$ is mapped to a (nonsimultaneous) region of size $\gamma_0\mathrm{d}^3\mathbf{x}$ (where γ_0 is computed from the boost velocity), and that $\mathrm{d}^3\mathbf{p}$ is shrunk by a factor of γ because it is transformed to a region where $d\gamma=0$ (because $\gamma\mathrm{d}\gamma=\mathbf{p}\cdot\mathrm{d}\mathbf{p}$ (because $\gamma^2=1+\mathbf{p}\cdot\mathbf{p}$) and $\mathbf{p}_0'=0$ post-transformation). Again, projecting onto simultaneous points does not change $\mathrm{d}^3\mathbf{x}$. In summary, we have argued that $\gamma_0\mathrm{d}^3\mathbf{x}$ and $\mathrm{d}^3\mathbf{p}/\gamma_0$ are scalars, and therefore their product $\mathrm{d}^3\mathbf{x}\wedge\mathrm{d}^3\mathbf{p}$ is a scalar.

2. *Lorentz-invariant form*

Observe that $\partial_t f_\mathrm{s}+\nabla_\mathbf{x}\cdot(\mathbf{v}f_\mathrm{s})=\partial_{\mathbf{x}^\mu}(\mathbf{v}^\mu f_\mathrm{s})$, which would be Lorentz-invariant if we replaced $\mathbf{v}^\mu$ with $\gamma\mathbf{v}^\mu=\mathbf{p}^\mu$. This indicates that to put the Vlasov equation (1) in a manifestly Lorentz-invariant form we need to multiply it by γ. Then we have

$$\partial_{\mathbf{x}^\mu}(\mathbf{p}^\mu f_\mathrm{s})+\gamma\nabla_{\mathbf{p}^k}(\mathbf{a}_\mathrm{s}^k f_\mathrm{s})=0;$$

the first term is manifestly Lorentz-invariant; for the second term we rewrite $\nabla_{\mathbf{p}^k}$ in terms of derivatives with respect to the four-vector $\mathbf{p}^\mu$ by regarding $\gamma=\mathbf{p}^0$ and $\mathbf{p}^k$ as independent quantities (that is, we extend the definition of all quantities (arbitrarily) beyond the quadratic manifold $\gamma^2=1+\mathbf{p}^2$). Then the chain rule for partial derivatives says

$$\partial_{\mathbf{p}^k}\mapsto\frac{\partial\gamma}{\partial\mathbf{p}^k}\partial_\gamma+\partial_{\mathbf{p}^k}=\frac{\mathbf{p}^k}{\gamma}\partial_\gamma+\partial_{\mathbf{p}^k},$$

so

$$\gamma\nabla_{\mathbf{p}^k}(\mathbf{a}_\mathrm{s}^k f_\mathrm{s})\mapsto\gamma\left(\frac{\mathbf{p}^k}{\gamma}\partial_\gamma(\mathbf{a}_\mathrm{s}^k f_\mathrm{s})+\nabla_{\mathbf{p}^k}(\mathbf{a}_\mathrm{s}^k f_\mathrm{s})\right).$$

Let $\widetilde{\mathbf{a}}_\mathrm{s}:=d_\tau\mathbf{p}=\gamma\mathbf{a}_\mathrm{s}$ be the proper acceleration. Then $\mathbf{p}\cdot\widetilde{\mathbf{a}}=\mathbf{p}\cdot d_\tau\mathbf{p}=\gamma d_\tau\gamma=\gamma\widetilde{\mathbf{a}}^0$, that is, $\mathbf{p}\cdot\mathbf{a}_\mathrm{s}=\widetilde{\mathbf{a}}^0$, so the acceleration term can indeed be cast into the form appearing in the following manifestly Lorentz-invariant version of the Vlasov equation,

$$\partial_{\mathbf{x}^\mu}(\mathbf{p}^\mu f_\mathrm{s})+\partial_{\mathbf{p}^\mu}(\widetilde{\mathbf{a}}_\mathrm{s}^\mu f_\mathrm{s})=0;$$

This confirms that the Vlasov equation (1) is indeed physical and Lorentz-invariant.

B. Moments

Let $\chi(\mathbf{p})$ be a generic moment to evaluate:

$$\int_\mathbf{p}\chi\Big(\partial_{\mathbf{x}^\mu}(\mathbf{p}^\mu f_\mathrm{s})+\gamma\nabla_{\mathbf{p}^k}(\mathbf{a}_\mathrm{s}^k f_\mathrm{s})=C_\mathrm{s}\Big)\frac{\mathrm{d}^3\mathbf{p}}{\gamma}.$$

Integrate by parts to get

$$\partial_{\mathbf{x}^\mu}\Big(\int_\mathbf{p}\chi\mathbf{p}^\mu f_\mathrm{s}\frac{\mathrm{d}^3\mathbf{p}}{\gamma}\Big)=\int_\mathbf{p}f_\mathrm{s}\widetilde{\mathbf{a}}_\mathrm{s}^k\left(\nabla_{\mathbf{p}^k}\chi\right)\frac{\mathrm{d}^3\mathbf{p}}{\gamma}+\int_\mathbf{p}C_\mathrm{s}\chi\frac{\mathrm{d}^3\mathbf{p}}{\gamma}.$$

Use $\mathbf{p}^{\otimes n}:=\otimes^n\mathbf{p}$ to denote the nth tensor power. In case $\chi=\mathbf{p}^{\otimes n}=\mathrm{Sym}\left(\mathbf{p}^{\otimes n}\right)$, we have (at least for spatial indices, and then by tensoriality of the expression for temporal indices as well): $\widetilde{\mathbf{a}}_\mathrm{s}^k\nabla_{\mathbf{p}^k}\chi=n\,\mathrm{Sym}\left(\mathbf{p}^{\otimes n-1}\widetilde{\mathbf{a}}_\mathrm{s}\right)=\frac{q_\mathrm{s}}{m_\mathrm{s}}n\,\mathrm{Sym}\left(\mathcal{F}\bullet\mathbf{p}^{\otimes n}\right)$, where I adopt the notation $(\mathcal{F}\bullet\mathbf{p})^\mu=\mathcal{F}^{\mu\nu}\mathbf{p}_\nu$ and use that $\widetilde{\mathbf{a}}_\mathrm{s}=\frac{q_\mathrm{s}}{m_\mathrm{s}}\mathcal{F}\bullet\mathbf{p}$, where $\mathcal{F}$ is the electromagnetic four-tensor with components

$$\mathcal{F}^{\mu\nu}:=\begin{bmatrix}0 & -\mathbf{E}^T/c\\ \mathbf{E}/c & \mathbf{B}\times\mathbb{I}\end{bmatrix}.$$

So for $\chi=\mathbf{p}^{\otimes n}$ $(n\geq 1)$ we have

$$\begin{aligned}\partial_{\mathbf{x}^\mu}\Big(\int_\mathbf{p}f_\mathrm{s}\mathbf{p}^{\otimes n}\mathbf{p}^\mu\frac{\mathrm{d}^3\mathbf{p}}{\gamma}\Big)=&\frac{q_\mathrm{s}}{m_\mathrm{s}}n\,\mathrm{Sym}\left(\mathcal{F}\bullet\int_\mathbf{p}f_\mathrm{s}\mathbf{p}^{\otimes n}\frac{\mathrm{d}^3\mathbf{p}}{\gamma}\right)\\ &+\int_\mathbf{p}C_\mathrm{s}\mathbf{p}^{\otimes n}\frac{\mathrm{d}^3\mathbf{p}}{\gamma}.\end{aligned}$$

For $\chi=1$ we get balance of particle four-flow:

$$\partial_{\mathbf{x}^\mu}\Big(\int_\mathbf{p}\mathbf{p}^\mu f_\mathrm{s}\frac{\mathrm{d}^3\mathbf{p}}{\gamma}\Big)=\int_\mathbf{p}C_\mathrm{s}\frac{\mathrm{d}^3\mathbf{p}}{\gamma}.$$

Separating time and space derivatives,

$$\partial_{ct}\Big(\int_\mathbf{p}f_\mathrm{s}\Big)+\nabla\cdot\Big(\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\Big)=\int_\mathbf{p}C_\mathrm{s}\frac{\mathrm{d}^3\mathbf{p}}{\gamma}.$$

More generally, for $\chi=\mathbf{p}^{\otimes n}$, separating time and space derivatives reveals a moment hierarchy:

$$\begin{aligned}&\partial_{ct}\int_\mathbf{p}f_\mathrm{s}\chi+\nabla\cdot\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\chi\\ &=\frac{q_\mathrm{s}}{m_\mathrm{s}}n\,\mathrm{Sym}\left(\mathcal{F}\bullet\int_\mathbf{p}f_\mathrm{s}\frac{\chi}{\gamma}\right)+\int_\mathbf{p}C_\mathrm{s}\frac{\chi}{\gamma}.\end{aligned}$$

Explicitly, omitting the collision term, the moment hierarchy is:

$$\begin{aligned}\chi:&\partial_{ct}\int_\mathbf{p}f_\mathrm{s}\chi+\nabla\cdot\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\chi=\frac{q_\mathrm{s}}{m_\mathrm{s}}n\,\mathrm{Sym}\left(\mathcal{F}\bullet\int_\mathbf{p}f_\mathrm{s}\frac{\chi}{\gamma}\right)\\ 1:&\ \partial_{ct}\Big(\int_\mathbf{p}f_\mathrm{s}\Big)+\nabla\cdot\Big(\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\Big)=0,\\ \mathbf{p}:&\ \partial_{ct}\int_\mathbf{p}f_\mathrm{s}\mathbf{p}+\nabla\cdot\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\mathbf{p}=\frac{q_\mathrm{s}}{m_\mathrm{s}}\Big(\mathcal{F}\bullet\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\Big),\\ \mathbf{p}\mathbf{p}:&\ \partial_{ct}\int_\mathbf{p}f_\mathrm{s}\mathbf{p}\mathbf{p}+\nabla\cdot\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\mathbf{p}\mathbf{p}=\frac{q_\mathrm{s}}{m_\mathrm{s}}2\,\mathrm{Sym}\left(\mathcal{F}\bullet\int_\mathbf{p}f_\mathrm{s}\mathbf{v}\mathbf{p}\right),\\ &\ldots\end{aligned}$$

Note that (1) the evolved moments are not tensorial (because $\frac{d\mathbf{p}}{\gamma}$, not $d\mathbf{p}$, is tensorial) and (2) the fluxes are not identical to the evolved moments (so closures are necessary for all the flux moments and not just the highest flux moment).

To separate space and time components of vectors, use that $\mathbf{v} = (1, \underline{v})$ and that $\mathbf{p} = (\gamma, \underline{p})$; then the moment hierarchy is:

$$
\begin{aligned}
\underline{p}^{\otimes n} &: \partial_{ct}\textstyle\int_{\mathbf{p}}\underline{p}^{\otimes n} f_{\mathrm{s}} + \nabla\cdot\int_{\mathbf{p}}\underline{v}\underline{p}^{\otimes n} f_{\mathrm{s}} = \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} n \,\mathrm{Sym}\left(c^{-1}\mathbf{E}\int_{\mathbf{p}}\underline{p}^{\otimes n-1} f_{\mathrm{s}} + \int_{\mathbf{p}}\underline{v}\underline{p}^{\otimes n-1} f_{\mathrm{s}} \times \mathbf{B}\right)\\
\gamma\underline{p}^{\otimes n-1} &: \partial_{ct}\textstyle\int_{\mathbf{p}}\gamma\underline{p}^{\otimes n-1} f_{\mathrm{s}} + \nabla\cdot\int_{\mathbf{p}}\underline{p}^{\otimes n} f_{\mathrm{s}} = \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} n \,\mathrm{Sym}\left(c^{-1}\mathbf{E}\cdot\int_{\mathbf{p}}\underline{v}\underline{p}^{\otimes n-1} f_{\mathrm{s}}\right)\\
1 &: \partial_{ct}\textstyle\int_{\mathbf{p}} f_{\mathrm{s}} + \nabla\cdot\int_{\mathbf{p}}\underline{v} f_{\mathrm{s}} = 0,\\
\underline{p} &: \partial_{ct}\textstyle\int_{\mathbf{p}}\underline{p} f_{\mathrm{s}} + \nabla\cdot\int_{\mathbf{p}}\underline{v}\underline{p} f_{\mathrm{s}} = \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}}\left(c^{-1}\mathbf{E}\int_{\mathbf{p}} f_{\mathrm{s}} + \int_{\mathbf{p}}\underline{v} f_{\mathrm{s}} \times \mathbf{B}\right),\\
\gamma &: \partial_{ct}\textstyle\int_{\mathbf{p}}\gamma f_{\mathrm{s}} + \nabla\cdot\int_{\mathbf{p}}\underline{p} f_{\mathrm{s}} = \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} c^{-1}\mathbf{E}\cdot\int_{\mathbf{p}}\underline{v} f_{\mathrm{s}},\\
\underline{pp} &: \partial_{ct}\textstyle\int_{\mathbf{p}} f_{\mathrm{s}}\underline{pp} + \nabla\cdot\int_{\mathbf{p}} f_{\mathrm{s}}\underline{vpp} = \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} 2\,\mathrm{Sym}\left(c^{-1}\mathbf{E}\cdot\int_{\mathbf{p}}\mathbf{p} + \int_{\mathbf{p}}\underline{vp} f_{\mathrm{s}} \times \mathbf{B}\right),\\
\gamma\underline{p} &: \partial_{ct}\textstyle\int_{\mathbf{p}}\gamma\underline{p} f_{\mathrm{s}} + \nabla\cdot\int_{\mathbf{p}}\underline{pp} f_{\mathrm{s}} = \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} 2\,\mathrm{Sym}\left(c^{-1}\mathbf{E}\cdot\int_{\mathbf{p}}\underline{vp} f_{\mathrm{s}}\right)\\
&\ldots
\end{aligned}
$$

C. Moment closure

For this section, assume that $c = 1$. For isotropic gas dynamics, one posits that there exists a reference frame in which the distribution f_{s} is isotropic. In this reference frame, the invariant flux tensor for $\chi = 1$ has a single nonzero component:

$$\textstyle\int_{\mathbf{p}}[1, \underline{v}] f_{\mathrm{s}} = [\rho, \underline{0}]$$

and the invariant flux tensor for $\chi = \mathbf{p}$ is diagonal:

$$\int_{\mathbf{p}} \begin{bmatrix} \gamma & \underline{p} \\ \underline{p} & \underline{vp} \end{bmatrix} f_{\mathrm{s}} = \begin{bmatrix} \mathcal{E} & \underline{0} \\ \underline{0} & \mathbb{I}\mathcal{P} \end{bmatrix};$$

the pressure $\mathcal{P}$ is related to the energy $\mathcal{E}$ and density ρ by an equation of state that is determined by the assumed distribution of particle velocities; usually the assumed distribution is defined to maximize some notion of entropy. To see how the assumed distribution relates pressure to energy, observe that pressure as well as energy is given by integrating the assumed distribution against a function of γ:

$$
\begin{aligned}
\textstyle\int_{\mathbf{p}}\underline{vp} f_{\mathrm{s}} &= \mathbb{I}\textstyle\int_{\mathbf{p}}\gamma^{-1}\underline{pp} f_{\mathrm{s}}\\
&= \mathbb{I}\textstyle\int_{\mathbf{p}}\gamma^{-1}\frac{1}{3}|\underline{p}|^2 f_{\mathrm{s}}\\
&= \mathbb{I}\displaystyle\int_{\mathbf{p}}\frac{\gamma^2 - 1}{3\gamma} f_{\mathrm{s}}.
\end{aligned}
$$

Moveover, by scalability in ρ, the equation of state reduces to a function of a single variable: $\frac{\mathcal{P}}{\rho} = \widetilde{\mathcal{P}}(\frac{\mathcal{E}}{\rho})$.

Determining values of ρ, $\mathcal{E}$, and the three velocity components of the Lorentz boost so as to match the five evolved moments is a nonlinear problem that is usually solved iteratively.
To determine the form of this system, we apply a Lorentz boost Λ that transforms from quantities in the reference frame of the fluid to a laboratory frame moving at proper velocity $\underline{U}$

$$\Lambda = \begin{bmatrix} \Gamma & \underline{U} \\ \underline{U} & \mathbb{I} + \frac{\underline{UU}}{\Gamma+1} \end{bmatrix},$$

where $\Gamma^2 = 1 + |\underline{U}|^2$.

Applying this boost to the tensor for the density and flux of mass says that

$$
\begin{aligned}
&\Lambda\cdot\begin{bmatrix} \rho \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \Gamma\rho \\ \underline{U}\rho \end{bmatrix}, \text{i.e.,}\\
&\rho' = \Gamma\rho,
\end{aligned}
$$

where ρ' is mass density in laboratory coordinates.

Applying this boost to the tensor for the density and flux of energy and momentum says that

$$\Lambda\cdot\begin{bmatrix} \mathcal{E} & \underline{0} \\ \underline{0} & \mathbb{I}\mathcal{P} \end{bmatrix}; \cdot\Lambda = \begin{bmatrix} \Gamma^2\mathcal{E} + |\underline{U}|^2\mathcal{P} & (\mathcal{E}+\mathcal{P})\Gamma\underline{U} \\ (\mathcal{E}+\mathcal{P})\Gamma\underline{U} & \mathcal{P}(\mathbb{I} + \underline{UU}) \end{bmatrix};$$