Nondimensionalization of plasma equations

by E. Alec Johnson, February 2011

Physical constants that define an ion-electron plasma are:

1. e, the magnitude of the charge of an electron,
2. m_i, m_e, the ion and electron mass, and
3. c, the speed of light.

Three fundamental parameters that characterize the state of a plasma are:

1. n_0, a typical particle density,
2. T_0, a typical temperature (often per species), and
3. B_0, a typical magnetic field strength.

In quasineutral equilibrium we can take $n_0 = n_i = n_e$ and $T_0 = T_i = T_e$. The thermal pressure is $p_0 := n_0 T_0$ and the magnetic pressure is $p_B := \frac{B_0^2}{2\mu_0}$.

Subsidiary space, time, and velocity scale parameters derived from the fundamental parameters are the gyrofrequencies $\omega_{g,s}$, the plasma frequencies $\omega_{p,s}$, the thermal velocity $v_{t,s}$, the Alfvén velocities $v_{A,s}$, the gyroradii $r_{g,s}$, the Debye length λ_D, and the inertial lengths (i.e. skin depths) δ_s:

1. $\omega_{g,s} = \frac{eB_0}{m_s}$,
2. $\omega_{p,s}^2 = \frac{n_0 e^2}{\epsilon_0 m_s}$,
3. $v_{t,s}^2 = \frac{T_s}{m_s} = \frac{p_s}{\rho_s}$, $\tilde{v}_{t,s}^2 := 2v_{t,s}^2$,
4. $v_{A,s}^2 = \frac{B_0^2}{\mu_0 m_s n_0} = \frac{2p_B}{\rho_s}$,
5. $r_{g,s} = \frac{v_{t,s}}{\omega_{g,s}} = \frac{m_s v_{t,s}}{eB_0}$, $\tilde{r}_{g,s} = \frac{\tilde{v}_{t,s}}{\omega_{g,s}}$,
6. $\lambda_D^2 = \left(\frac{v_{t,s}}{\omega_{p,s}}\right)^2 = \frac{\epsilon_0 T_0}{n_0 e^2}$,
7. $\delta_s^2 = \left(\frac{c}{\omega_{p,s}}\right)^2 = \left(\frac{v_{A,s}}{\omega_{g,s}}\right)^2 = \frac{m_s}{\mu_0 n_s e^2}$

Note that (most?) often in the literature the thermal velocity is defined as $\tilde{v}_{t,s}$ rather than $v_{t,s}$. We say that two parameters are *equivalent* if one is a constant multiple of the other. For example, the thermal velocities are equivalent to one another and to the sound speed $\sqrt{\frac{\gamma p_0}{\rho_0}}$. Important nondimensional ratios are the plasma beta $\beta := \frac{p_0}{p_B}$ and the ratio of the speed of light to the Alfvén speed. Other nondimensional ratios can be defined in terms of these ratios:

1. $\beta = \frac{p_0}{p_B} = \left(\frac{\tilde{v}_{t,s}}{v_{A,s}}\right)^2 = \left(\frac{\tilde{r}_{g,s}}{\delta_s}\right)^2$,
2. $\frac{c}{v_{A,s}} = \frac{r_{g,s}}{\lambda_D} = \frac{\omega_{p,s}}{\omega_{g,s}}$.

The subsidiary parameters (except for the temperature-related parameters $v_{t,s}$ and λ_D) emerge from a generic nondimensionalization of the particle (or Vlasov or 2-fluid) equations.

Choose values for:

t_0	(time scale)	(e.g. ion gyroperiod $1/\omega_{g,i}$),
x_0	(space scale)	(e.g. ion skin depth δ_i),
m_0	(mass scale)	(e.g. ion mass m_i),
$e = q_0$	(charge scale)	(e.g. ion charge e),
B_0	(magnetic field)	(e.g. $\omega_{g,i} m_i / e$), and
n_0	(number density)	(e.g. something $\gg 1/x_0^3$).

This implies typical values for:

$v_0 = x_0/t_0$	(velocity),
$E_0 = B_0 v_0$	(electric field),
$\sigma_0 = e n_0$	(charge density),
$J_0 = e n_0 v_0$	(current density), and
$S_0 = n_0$	(no. particles per unit number density).

Making the substitutions

$$
\begin{aligned}
t &= \tilde{t} t_0, & \mathbf{E} &= \widetilde{\mathbf{E}} B_0 v_0, \\
\mathbf{x} &= \widetilde{\mathbf{x}} x_0, & \sigma &= \widetilde{\sigma} e n_0, \\
q &= \widetilde{q} e, & \mathbf{J} &= \widetilde{\mathbf{J}} e n_0 v_0, \\
m &= \widetilde{m} m_0, & S_p(\mathbf{x}_p) &= \widetilde{S}_p(\widetilde{\mathbf{x}}_p) n_0, \\
n &= \widetilde{n} n_0, & c &= \widetilde{c} v_0, \\
\mathbf{B} &= \widetilde{\mathbf{B}} B_0, & \mathbf{v} &= \widetilde{\mathbf{v}} v_0
\end{aligned}
$$

in the fundamental equations

$$
\begin{aligned}
\partial_t \mathbf{B} &= -\nabla_{\mathbf{x}} \times \mathbf{E}, & \nabla_{\mathbf{x}} \cdot \mathbf{B} &= 0, \\
\partial_t \mathbf{E} &= c^2 \nabla_{\mathbf{x}} \times \mathbf{B} - \mathbf{J}/\epsilon_0, & \nabla_{\mathbf{x}} \cdot \mathbf{E} &= \sigma/\epsilon_0, \\
\mathbf{J} &= \sum_p S_p(\mathbf{x}_p) q_p v_p, & \sigma &= \sum_p S_p(\mathbf{x}_p) q_p,
\end{aligned}
$$

and

$$d_t(\gamma\mathbf{v}_p) = \frac{q_p}{m_p}\Big(\mathbf{E}(\mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(\mathbf{x}_p)\Big),$$
$$d_t\mathbf{x}_p = \mathbf{v}_p$$

gives the almost identical-appearing nondimensionalized system

$$\begin{aligned}
&\partial_{\tilde{t}}\widetilde{\mathbf{B}} = -\nabla_{\tilde{\mathbf{x}}}\times\widetilde{\mathbf{E}}, && \nabla_{\tilde{\mathbf{x}}}\cdot\widetilde{\mathbf{B}} = 0,\\
&\partial_{\tilde{t}}\widetilde{\mathbf{E}} = \tilde{c}^2\nabla_{\tilde{\mathbf{x}}}\times\widetilde{\mathbf{B}} - \widetilde{\mathbf{J}}/\varepsilon, && \nabla_{\tilde{\mathbf{x}}}\cdot\widetilde{\mathbf{E}} = \tilde{\sigma}/\varepsilon,\\
&\widetilde{\mathbf{J}} = \sum_p \widetilde{S}_p(\tilde{\mathbf{x}}_p)\tilde{q}_p\tilde{\mathbf{v}}_p, && \tilde{\sigma} = \sum_p \widetilde{S}_p(\tilde{\mathbf{x}}_p)\tilde{q}_p,
\end{aligned}$$

and

$$d_{\tilde{t}}(\gamma\tilde{\mathbf{v}}_p) = (t_0\omega_g)\frac{\tilde{q}_p}{\tilde{m}_p}\Big(\widetilde{\mathbf{E}}(\tilde{\mathbf{x}}_p) + \tilde{\mathbf{v}}_p\times\widetilde{\mathbf{B}}(\tilde{\mathbf{x}}_p)\Big),$$
$$d_{\tilde{t}}\tilde{\mathbf{x}}_p = \tilde{\mathbf{v}}_p;$$

here $t_0\omega_g = t_0\frac{q_0B_0}{m_0}$ is the gyrofrequency nondimensionalized by a choice of t_0 (which can be chosen to be the gyroperiod in order to set this factor to unity) and $\frac{1}{\varepsilon} = \frac{x_0n_0e}{v_0B_0\varepsilon_0} = t_0\frac{eB_0}{m_0}\frac{\mu_0m_0n_0}{B_0^2}c^2 = (t_0\omega_g)\left(\frac{c}{v_A}\right)^2$. Note that we can also write $(t_0\omega_g) = \frac{x_0}{r_g}$.

1 Collisional nondimensional ratios

Up to this point we have neglected collisional effects. What happens if we nondimensionalize a collisional plasma model? Our starting point for collisional plasma models is the Boltzmann equation, which asserts that the density $f_s(t,\mathbf{x},\mathbf{v})$ of particles of species s in phase space is conserved.

$$\partial_t f_s + \mathbf{v}\cdot\nabla_{\mathbf{x}}f_s + \mathbf{a}\cdot\nabla_{\mathbf{v}}f_s = C_s$$

The simplest collision operator is relaxation toward a Maxwellian:

$$C_s = (f_{M,s} - f_s)/\tau_s,$$

where $f_{M,s}$ is a Maxwellian distribution (with the same number density, temperature, and velocity) and τ_s is a relaxation period (typically the isotropization period). The Braginskii formula for the relaxation period is

$$\tau_s = \tau_b\frac{\sqrt{m_s}T_s^{3/2}}{n}$$

where (for electrons) the base isotropization rate is

$$\tau_b = \frac{3(2\pi)^{3/2}\varepsilon_0^2}{e^4\ln\Lambda},$$

where $\ln\Lambda$, the Coulomb logarithm, is typically between 10 and 20, and for electrons is given by the formula

$$\Lambda = 12\pi\frac{n_e}{Z}\left(\frac{\varepsilon_0T_e}{n_ee^2}\right)^{3/2};$$

Λ is on the order of the number of particles in a Debye sphere.

The viscosity μ_s and heat flux K_s are related to the relaxation period (up to a fixed constant of order 1) by the relations

$$\mu_s = \tau_sp_s \quad = \tau_b\sqrt{m_s}T_s^{5/2},$$
$$K_s = \mu_s/m_s = \tau_b\frac{T_s^{5/2}}{\sqrt{m_s}}.$$

1.1 MHD

Ideal MHD takes $c\to\infty$ and $r\to 0$ and therefore has only one free parameter (μ_0) rather than three. To nondimensionalize the MHD equations we make the substitution

$$\mathbf{B} = \sqrt{\mu_0}\hat{\mathbf{B}};$$

this eliminates μ_0. More generically, we can nondimensionalize using a typical Alfvén speed, density, and time scale:

$$\begin{aligned}
&v_A = \frac{B_0}{\sqrt{\mu_0\rho_0}}, && \mathbf{B} = B_0\hat{\mathbf{B}}, && \rho = \rho_0\hat{\rho}, && t = t_0\hat{t},\\
&\mathbf{u} = v_A\hat{\mathbf{u}}, && p = \rho_0v_A^2\hat{p}, && \mathcal{E} = \rho_0v_A^2\hat{\mathcal{E}}, && x = v_At_0\hat{x}.
\end{aligned}$$