Waves in Maxwell's Equations

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1 Light waves

Recall Maxwell's equations in a vacuum:

$$\begin{aligned}
&\partial_t B + c_1 \nabla\times\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0,\\
&\partial_t E - c_2 \nabla\times\mathbf{B} = 0, \quad \nabla\cdot\mathbf{E} = 0.
\end{aligned}$$

For SI units $c_1 = 1$ and $c_2 = c^2$; for Gaussian units, $c_1 = c$ and $c_2 = c$.

In one dimension this becomes two decoupled systems:

$$\begin{aligned}
&\partial_t \begin{pmatrix} B_y \\ E_z \end{pmatrix} - \partial_x \begin{pmatrix} c_1 E_z \\ c_2 B_y \end{pmatrix} = 0,\\
&\partial_t \begin{pmatrix} B_z \\ E_y \end{pmatrix} + \partial_x \begin{pmatrix} c_1 E_y \\ c_2 B_z \end{pmatrix} = 0.
\end{aligned}$$

In matrix form these read:

$$\begin{aligned}
&\begin{pmatrix} B_y \\ E_z \end{pmatrix}_t - \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} B_y \\ E_z \end{pmatrix}_x = 0,\\
&\begin{pmatrix} B_z \\ E_y \end{pmatrix}_t + \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} B_z \\ E_y \end{pmatrix}_x = 0,
\end{aligned}$$

To find the eigenstructure, we row reduce the systems

$$\begin{aligned}
&\begin{pmatrix} c & -c_1 \\ -c_2 & c \end{pmatrix} \cdot \begin{pmatrix} B_y \\ E_z \end{pmatrix}' = 0,\\
&\begin{pmatrix} c & c_1 \\ c_2 & c \end{pmatrix} \cdot \begin{pmatrix} B_z \\ E_y \end{pmatrix}' = 0.
\end{aligned}$$

The eigenvalues are

$$c = \pm c_0, \quad \text{where} \quad c_0 := \sqrt{c_1 c_2}.$$

Left and right eigenvectors for $c = \pm c_0$ are

$$\begin{aligned}
&\begin{pmatrix} B_y \\ E_z \end{pmatrix}'_{\text{right}} = \begin{pmatrix} \mp 1 \\ \sqrt{\frac{c_2}{c_1}} \end{pmatrix}, \quad \begin{pmatrix} B_y \\ E_z \end{pmatrix}'_{\text{left}} = \frac{1}{2}\begin{pmatrix} \mp 1 \\ \sqrt{\frac{c_1}{c_2}} \end{pmatrix},\\
&\begin{pmatrix} B_z \\ E_y \end{pmatrix}'_{\text{right}} = \begin{pmatrix} \pm 1 \\ \sqrt{\frac{c_2}{c_1}} \end{pmatrix}, \quad \begin{pmatrix} B_z \\ E_y \end{pmatrix}'_{\text{left}} = \frac{1}{2}\begin{pmatrix} \pm 1 \\ \sqrt{\frac{c_1}{c_2}} \end{pmatrix}.
\end{aligned}$$

That is, the $R\Lambda L$ diagonalization is

$$\begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \sqrt{\frac{c_2}{c_1}} & \sqrt{\frac{c_2}{c_1}} \end{pmatrix} \begin{pmatrix} -c_0 & \\ & c_0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & \sqrt{\frac{c_1}{c_2}} \\ 1 & \sqrt{\frac{c_1}{c_2}} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -c_1 \\ -c_2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \sqrt{\frac{c_2}{c_1}} & \sqrt{\frac{c_2}{c_1}} \end{pmatrix} \begin{pmatrix} -c_0 & \\ & c_0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & \sqrt{\frac{c_1}{c_2}} \\ -1 & \sqrt{\frac{c_1}{c_2}} \end{pmatrix}.$$

2 Light waves with correction potentials

Following [1], to attempt to enforce the divergence constraints we can use correction potentials.

$$\begin{aligned}
&\partial_t \begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix} + \begin{bmatrix} c_1 \nabla\times\mathbf{E} + b_1 \nabla\psi \\ -c_2 \nabla\times\mathbf{B} + b_2 \nabla\phi \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathbf{J}/\varepsilon \end{bmatrix},\\
&\partial_t \begin{bmatrix} \psi \\ \phi \end{bmatrix} + \begin{bmatrix} a_1 \nabla\cdot\mathbf{B} \\ a_2 \nabla\cdot\mathbf{E} \end{bmatrix} = \begin{bmatrix} 0 \\ a_2 \sigma/\varepsilon \end{bmatrix} - \begin{bmatrix} \varepsilon_1 \psi \\ \varepsilon_2 \phi \end{bmatrix}.
\end{aligned}$$

The correction potentials ψ and ϕ are for numerical divergence cleaning purposes. Taking the divergence of the evolution equation for $\mathbf{B}$ gives the system

$$\partial_t \begin{bmatrix} \nabla\cdot\mathbf{B} \\ \psi \end{bmatrix} + \begin{bmatrix} b_1 \nabla^2 \psi \\ a_1 \nabla\cdot\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -\varepsilon_1 \psi \end{bmatrix}.$$

To eliminate ψ take the Laplacian of the second equation and get a telegraph equation for $\nabla\cdot\mathbf{B}$:

$$\partial_{tt} \nabla\cdot\mathbf{B} - b_1 a_1 \nabla^2 \nabla\cdot\mathbf{B} + \varepsilon_1 \partial_t \nabla\cdot\mathbf{B} = 0.$$

To eliminate $\mathbf{B}$ take the time derivative of the second equation and get a telegraph equation for ψ:

$$\partial_{tt} \psi - b_1 a_1 \nabla^2 \psi + \varepsilon_1 \partial_t \psi = 0.$$

Taking the divergence of the evolution equation for $\mathbf{E}$ and using $\partial_t \sigma + \nabla\cdot\mathbf{J} = 0$ gives the system

$$\partial_t \begin{bmatrix} (\nabla\cdot\mathbf{E} - \sigma/\varepsilon) \\ \phi \end{bmatrix} + \begin{bmatrix} b_2 \nabla^2 \phi \\ a_2 (\nabla\cdot\mathbf{E} - \sigma/\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ -\varepsilon_2 \phi \end{bmatrix}.$$

This has the same form as for the magnetic field, so:

$$\begin{aligned}
&\left(\partial_{tt} - a_2 b_2 \nabla^2 + \varepsilon_2 b_2 \partial_t\right)(\nabla\cdot\mathbf{E} - \sigma/\varepsilon) = 0,\\
&\left(\partial_{tt} - a_2 b_2 \nabla^2 + \varepsilon_2 \partial_t\right)\phi = 0.
\end{aligned}$$

3 Eigenstructure with correction potentials

With correction potentials Maxwell's equations in a vacuum assert

$$\partial_t \begin{bmatrix}\mathbf{B}\\ \mathbf{E}\end{bmatrix} + \begin{bmatrix} c_1\nabla\times\mathbf{E}+b_1\nabla\psi \\ -c_2\nabla\times\mathbf{B}+b_2\nabla\phi\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\partial_t \begin{bmatrix}\psi\\ \phi\end{bmatrix} + \begin{bmatrix} a_1\nabla\cdot\mathbf{B} \\ a_2\nabla\cdot\mathbf{E}\end{bmatrix} + \begin{bmatrix}\varepsilon_1\psi\\ \varepsilon_2\phi\end{bmatrix} = 0.$$

For SI units $c_1 = 1$ and $c_2 = c^2$; for Gaussian units, $c_1 = c$ and $c_2 = c$. Customarily $a_1 = a_2 = (\chi c)^2$ and $b_1 = b_2 = 1$.

In one dimension, ignoring the source terms, this becomes four decoupled systems:

$$\partial_t \begin{pmatrix}B_x\\ \psi\end{pmatrix} + \partial_x \begin{pmatrix}b_1\psi\\ a_1B_x\end{pmatrix} = 0,$$

$$\partial_t \begin{pmatrix}E_x\\ \phi\end{pmatrix} + \partial_x \begin{pmatrix}b_2\phi\\ a_2E_x\end{pmatrix} = 0,$$

$$\partial_t \begin{pmatrix}B_y\\ E_z\end{pmatrix} - \partial_x \begin{pmatrix}c_1E_z\\ c_2B_y\end{pmatrix} = 0,$$

$$\partial_t \begin{pmatrix}B_z\\ E_y\end{pmatrix} + \partial_x \begin{pmatrix}c_1E_y\\ c_2B_z\end{pmatrix} = 0.$$

In matrix form the correction potential systems read

$$\begin{pmatrix}B_x\\ \psi\end{pmatrix}_t + \begin{pmatrix}0 & b_1\\ a_1 & 0\end{pmatrix}\cdot\begin{pmatrix}B_x\\ \psi\end{pmatrix}_x = 0,$$

$$\begin{pmatrix}E_x\\ \phi\end{pmatrix}_t + \begin{pmatrix}0 & b_2\\ a_2 & 0\end{pmatrix}\cdot\begin{pmatrix}E_x\\ \phi\end{pmatrix}_x = 0.$$

To find the eigenstructure, we row reduce e.g.

$$\begin{pmatrix}c & b_1\\ a_1 & c\end{pmatrix}\cdot\begin{pmatrix}B_x\\ \psi\end{pmatrix}' = 0.$$

The eigenvalues are

$$c_1 = \pm\sqrt{b_1a_1} = \pm\chi c.$$

Corresponding left and right eigenvectors are

$$\begin{pmatrix}B_x\\ \psi\end{pmatrix}'_{\text{right}} = \begin{pmatrix}\pm 1\\ \sqrt{\frac{a_1}{b_1}}\end{pmatrix}, \quad \begin{pmatrix}B_x\\ \psi\end{pmatrix}'_{\text{left}} = \frac{1}{2}\begin{pmatrix}\pm 1\\ \sqrt{\frac{b_1}{a_1}}\end{pmatrix},$$

or customarily

$$\begin{pmatrix}B_x\\ \psi\end{pmatrix}'_{\text{right}} = \begin{pmatrix}\pm 1\\ \chi c\end{pmatrix}, \quad \begin{pmatrix}B_x\\ \psi\end{pmatrix}'_{\text{left}} = \frac{1}{\chi c}\begin{pmatrix}\pm\chi c\\ 1\end{pmatrix}.$$

That is, the $R\Lambda L$ diagonalization is

$$\begin{pmatrix}0 & 1\\ (\chi c)^2 & 0\end{pmatrix} = \begin{pmatrix}-1 & 1\\ \chi c & \chi c\end{pmatrix}\begin{pmatrix}-\chi c & \\ & \chi c\end{pmatrix}\frac{1}{2\chi c}\begin{pmatrix}-\chi c & 1\\ \chi c & 1\end{pmatrix},$$

or in general

$$\begin{pmatrix}0 & b_1\\ a_1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 1\\ \sqrt{\frac{a_1}{b_1}} & \sqrt{\frac{a_1}{b_1}}\end{pmatrix}\begin{pmatrix}-c_1 & \\ & c_1\end{pmatrix}\frac{1}{2}\begin{pmatrix}-1 & \sqrt{\frac{b_1}{a_1}}\\ 1 & \sqrt{\frac{b_1}{a_1}}\end{pmatrix},$$

4 Telegraph equation

Consider the telegraph equation

$$u_{tt} - a^2u_{xx} + 2\varepsilon a u_t = 0.$$

Seek a solution $u = e^{-rt}e^{ikx}$. Substituting gives

$$r^2 + a^2k^2 - 2\varepsilon a r = 0.$$

So

$$\begin{aligned} r &= a\varepsilon \pm \sqrt{(a\varepsilon)^2 - (ak)^2} \\ &= a\varepsilon\left(1 \pm \sqrt{1 - (k/_\varepsilon)^2}\right) \\ &= a\varepsilon\left(1 \pm i\sqrt{(k/_\varepsilon)^2 - 1}\right) \end{aligned}$$

For a given wavelength $\lambda = 2\pi/k$ the overall rate of decay is the minimum of the real parts of r:

$$r_0 =: \left\{\begin{array}{ll} a\varepsilon & \text{if } |k| >= \varepsilon, \\ a\varepsilon\left(1 - \sqrt{1-(k/_\varepsilon)^2}\right) & \text{if } |k| <= \varepsilon. \end{array}\right\}$$

Sketch r_0 as a function of $|k|$ (or $r_0/_{(a\varepsilon)}$ as a function of $|k|/_\varepsilon$). For $|k| > \varepsilon$ the rate of advection is $\omega/_{|k|} := a\sqrt{(|k|/_\varepsilon)^2 - 1/_{(|k|/_\varepsilon)}} \to a$ as $|k|/_\varepsilon \to \infty$. The peak rate of decay that wavelength k can experience is for $\varepsilon = |k|$. Higher values of ε neither damp this frequency effectively nor convect it. This suggests allowing ε to vary with time in order to disperse and damp all frequencies, e.g. $\varepsilon(t) = \varepsilon_0 \sin^2(a\varepsilon_0 t)$, where $a\varepsilon_0$ is the desired rate of damping of high frequencies. For the correction potentials what happens if we replace $\varepsilon_1\psi$ with some maybe nonlinear $f(\psi, \nabla\psi)$? The challenge is to damp low frequencies.

References

[1] A. Dedner and F. Kemm and D. Kröner and C.-D. Munz and T. Schnitzer and M. Wesenberg, *Hyperbolic divergence cleaning for the MHD equations*, J. Comp. Phys., 2002.