Waves in MHD

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1 Ideal MHD eigenstructure

1.1 Linearization

Recall the equations of smooth MHD in primitive variables:

$$\begin{aligned}
&\rho_{,t} + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&\rho(\mathbf{u}_{,t} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = \mu_0^{-1}(\nabla\times\mathbf{B})\times\mathbf{B},\\
&p_{,t} + \mathbf{u}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{u} = 0,\\
&\mathbf{B}_{,t} + \nabla\times(\mathbf{B}\times\mathbf{u}) = 0,
\end{aligned}$$

where we have used Ampere's law $\mathbf{J} = \mu_0\nabla\times\mathbf{B}$ and the comma notation is used for partial differentiation. We remark that the pressure evolution equation is a form of the thermal energy evolution equation, which is obtained from energy conservation by subtracting kinetic energy balance (obtained by dotting $\mathbf{u}$ with the momentum equation) and subtracting magnetic field energy balance (obtained by dotting $\mathbf{B}$ with the magnetic field evolution equation). It implies that entropy is invariant along particle paths.

To facilitate linearization, we apply the product rule and rewrite these equations in a form that contains no derivatives of products. (We vertically align according to differentiated variable to prepare to put the equations in matrix form.)

$$\begin{aligned}
0 &= \rho_{,t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u},\\
0 &= \mathbf{u}_{,t} \quad + \mathbf{u}\cdot\nabla\mathbf{u} \quad + \frac{1}{\rho}\nabla p \; + \frac{1}{\mu_0\rho}\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{\mu_0\rho}(\nabla\mathbf{B})\cdot\mathbf{B},\\
0 &= p_{,t} \quad + \gamma p\nabla\cdot\mathbf{u} \quad + \mathbf{u}\cdot\nabla p,\\
0 &= \mathbf{B}_{,t} \quad + \mathbf{B}\nabla\cdot\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{u} \quad + \mathbf{u}\cdot\nabla\mathbf{B}.
\end{aligned}$$

Assuming $\partial_2 = 0 = \partial_3$ implies that B_1 is constant (in space by the divergence condition $\nabla\cdot\mathbf{B} = 0$ and in time by Faraday's law $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$) and thus gives the 1-dimensional MHD system

$$\begin{pmatrix}\rho\\u_1\\u_2\\u_3\\p\\B_2\\B_3\end{pmatrix}_{,t} + \begin{bmatrix}
u_1 & \rho & 0 & 0 & 0 & 0 & 0\\
0 & u_1 & 0 & 0 & \frac{1}{\rho} & \frac{B_2}{\mu_0\rho} & \frac{B_3}{\mu_0\rho}\\
0 & 0 & u_1 & 0 & 0 & \frac{-B_1}{\mu_0\rho} & 0\\
0 & 0 & 0 & u_1 & 0 & 0 & \frac{-B_1}{\mu_0\rho}\\
0 & \gamma p & 0 & 0 & u_1 & 0 & 0\\
0 & B_2 & -B_1 & 0 & 0 & u_1 & 0\\
0 & B_3 & 0 & -B_1 & 0 & 0 & u_1
\end{bmatrix}\cdot\begin{pmatrix}\rho\\u_1\\u_2\\u_3\\p\\B_2\\B_3\end{pmatrix}_{,x} = 0$$

This is in the quasilinear form

$$U_{,t} + \mathbb{A}\cdot U_{,x} = 0.$$

To linearize, we merely freeze the matrix entries at $\mathbb{A}_0 := \mathbb{A}(U_0)$, where U_0 is a background state, and replace the differentiated state variables with perturbed versions:

$$\widetilde{U}_{,t} + \mathbb{A}_0\cdot\widetilde{U}_{,x} \approx 0, \tag{1}$$

where $\widetilde{U} := U - U_0$.

The eigenstructure of $\mathbb{A}_0$ reveals waves of the linearized system. Indeed, suppose that

$$\mathbb{A}_0\cdot U' = \lambda U'. \tag{2}$$

Then $\widetilde{U} = U' f(x-\lambda t)$ satisfies the linearized ODE (1) for any scalar-valued differentiable function f. Typically $f(x)$ is assumed to be $\sin(kx)$, $\cos(kx)$, or e^{ikx}.

Recall that to solve eigenproblem (3), we write it as the homogeneous linear problem

$$(\mathbb{A}_0 - \lambda\mathbb{I})\cdot U' = 0. \tag{3}$$

The eigenvalue λ is the wave speed, so we write it as $\lambda = u_1 + c$, where c represent the speed of the wave in the reference frame of the fluid.

1.2 Eigenstructure

To simplify notation and to facilitate finding left eigenvectors later, we generalize the matrix (and make it look closer to a "generically self-adjoint" matrix) by making the definitions

$$\begin{aligned}
g &:= \gamma p,\\
g^* &:= \frac{1}{\rho},\\
\mathbf{B}^* &:= \frac{\mathbf{B}}{\mu_0\rho}.
\end{aligned}$$

Observe that $gg^* = \frac{\gamma p}{\rho} =: v_s$, the acoustic sound speed. We remark that we could assume without loss of generality that $\rho = 1 = g^*$ by replacing $\mathbf{B}$ with $\sqrt{\mu_0^{-1}}\mathbf{B}$ and by choice of units of mass. Furthermore, if we are willing to rescale time (or space), we could also assume that $v_s = 1$ by choice of units of velocity.

Transforming into a frame of reference convected with the fluid shifts the diagonal entries to zero. By choosing units of mass appropriately, we can force $\rho_0 = 1 = g^*$. By choosing units of time properly, we can also force the sound speed to be one, i.e., $v_s = 1 = g$. The result of such a redimensionalization is to make the lower right 6×6 matrix symmetric. It easily follows that the matrix as a whole has real eigenvalues and a full set of eigenvectors. Thus the system is hyperbolic, and its general solution is a superposition of "eigenperturbations" propagating at a speed equal to the corresponding eigenvalue.

So our formally near-self-adjoint system is

$$\widetilde{\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}}_{,t} + \begin{bmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & g^* & B_2^* & B_3^* \\ 0 & 0 & u_1 & 0 & 0 & -B_1^* & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & -B_1^* \\ 0 & g & 0 & 0 & u_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & u_1 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & u_1 \end{bmatrix}_0 \cdot \widetilde{\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}}_{,x} = 0.$$

By Galilean relativity, we can assume without loss of generality that the background state satisfies $u_1 = 0$. By rotation of coordinates in dimensions 2 and 3, we may also assume that the background state satisfies $B_3 = 0$ and $B_2 \geq 0$.

So to find the eigenstructure, we row-reduce the system

$$\begin{bmatrix} -c & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & g^* & B_2^* & 0 \\ 0 & 0 & -c & 0 & 0 & -B_1^* & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & -B_1^* \\ 0 & g & 0 & 0 & -c & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & -B_1 & 0 & 0 & -c \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}' = 0;$$

avoiding division will give eigenvectors with polynomial components and a polynomial dispersion relation for the eigenvalues. The dispersion relation can then be used to rewrite any polynomial in c in terms of polynomials of lesser order than the order of the dispersion relation. This technique is useful in simplifing expressions for the norms of the eigenvectors.

This system decouples into the following subsystems:

$$\begin{bmatrix} c & B_1^* \\ B_1 & c \end{bmatrix}_0 \cdot \begin{pmatrix} u_3 \\ B_3 \end{pmatrix}' = 0$$

and

$$\begin{bmatrix} -c & 1 & 0 & 0 & 0 \\ 0 & -c & 0 & g^* & B_2^* \\ 0 & 0 & -c & 0 & -B_1^* \\ 0 & g & 0 & -c & 0 \\ 0 & B_2 & -B_1 & 0 & -c \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0.$$

The Alfvén system on the left gives rise to a pair of "oblique" Alfvén waves. The magnetosonic system on the right gives rise to an entropy wave and pairs of fast and slow magnetosonic waves.

The velocities of the oblique Alfvén waves are given by $c_A = \pm\sqrt{B_1 B_1^*} = \pm\frac{B_1}{\sqrt{\mu_0 \rho}}$, and the corresponding eigenvectors are

$$\begin{pmatrix} u_3 \\ B_3 \end{pmatrix}' \propto \begin{pmatrix} c \\ \mp B_1 \end{pmatrix} \propto \begin{pmatrix} 1 \\ \mp\sqrt{\mu_0 \rho} \end{pmatrix}$$

Notice that the pertubations are in the plane perpendicular to the direction of propagation.

To find the characteristic polynomial and right eigenvectors of the magnetosonic system, we row-reduce to upper triangular form. We leave out the first row and column for now, since nothing needs to be done there.

Our system is

$$\begin{bmatrix} -c & 0 & g^* & B_2^* \\ 0 & -c & 0 & -B_1^* \\ g & 0 & -c & 0 \\ B_2 & -B_1 & 0 & -c \end{bmatrix}_0 \cdot \begin{pmatrix} u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0,$$

Notice that the pertubations are in the plane spanned by the direction of propagation and the magnetic field. Notice also that just as for Alfvén waves, B_2' and u_2' (which are in the direction perpendicular to wave propagation) are in a ratio equal to the ratio of the wave speed, and that just as for sound waves, p' and u_1' (which are related to perturbations parallel to the direction of motion) are in a ratio equal to the ratio of g to the sound speed.

This system is similar to

$$\begin{bmatrix} g & 0 & -c & 0 \\ 0 & c & 0 & B_1^* \\ 0 & gB_1 & -cB_2 & cg \\ 0 & 0 & gg^* - c^2 & gB_2^* \end{bmatrix}_0 \cdot \begin{pmatrix} u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0,$$

which is similar to

$$\begin{bmatrix} g & 0 & -c & 0 \\ 0 & c & 0 & B_1 \\ 0 & 0 & c^2 B_2 & g(B_1 B_1^* - c^2) \\ 0 & 0 & gg^* - c^2 & gB_2^* \end{bmatrix}_0 \cdot \begin{pmatrix} u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0.$$

For the determinant to vanish, the determinant of the lower right 2-by-2 matrix must vanish, i.e., the last two equations must be redundant.

Taking the determinant of the lower right 2-by-2 system yields the dispersion relation

$$(c^2 - B_1 B_1^*)(c^2 - gg^*) - c^2 B_2 B_2^* = 0, \text{ i.e.,}$$
$$c^4 - c^2(gg^* + \mathbf{B} \cdot \mathbf{B}^*) + gg^* B_1 B_1^* = 0,$$

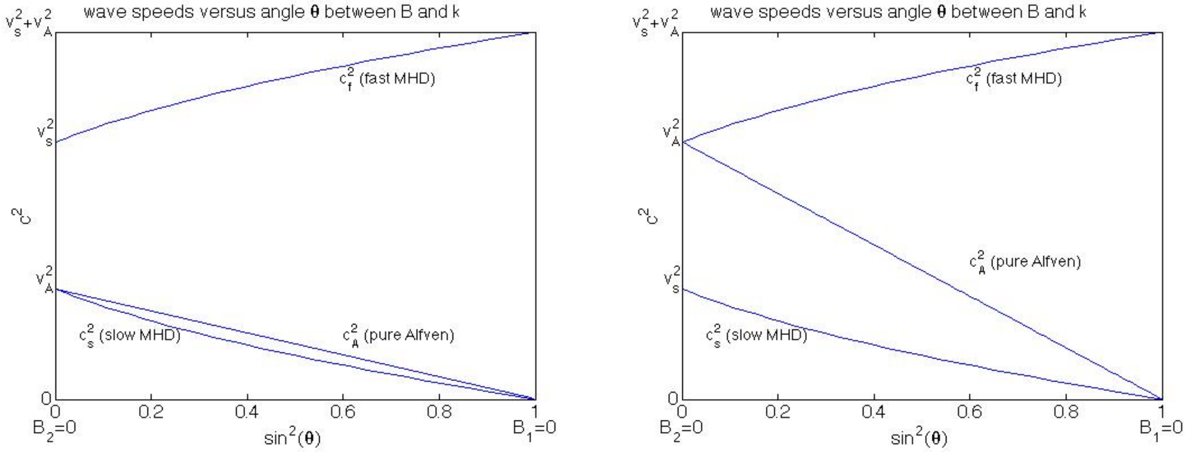
i.e.,

$$c^4 - c^2(v_s^2 + v_A^2) + v_s^2 c_A^2 = 0,$$

where

$$v_A^2 := \mathbf{B} \cdot \mathbf{B}^* = \frac{B^2}{\mu_0 \rho} = \text{ (Alfvén speed),}$$
$$c_A^2 := B_1 B_1^* = \frac{B_1^2}{\mu_0 \rho} = \text{ (oblique Alfvén wave speed), and}$$
$$v_s^2 := gg^* = \frac{\gamma p}{\rho} = \text{ (sound speed).}$$

Note that $c_A^2 = v_A^2 \cos^2\theta$, where θ is the angle between $\mathbf{B}$ and the positive x-axis.



(a) $v_A^2 < v_s^2$ (b) $v_A^2 > v_s^2$

Figure 1: MHD wave speeds as a function of the angle θ between the magnetic field **B** and the direction of propagation **k**.

The roots of the quadratic dispersion relation in c^2 define the fast and slow magnetosonic speeds:

$$(c^2 - c_f^2)(c^2 - c_s^2) = 0.$$

So $c_f^2, c_s^2 = (1/2)\left[(v_s^2 + v_A^2) \pm \sqrt{(v_s^2 + v_A^2)^2 - 4v_s^2 c_A^2}\right]$, an expression of which we make little direct use, preferring to work directly with the polynomial dispersion relation.

Observe that we can rewrite the magnetosonic dispersion relation as

$$(c^2 - v_s^2)(c^2 - v_A^2) = v_s^2 v_A^2 \sin^2\theta.$$

Recall also the formula for the Alfvén speed,

$$c_A^2 = v_A^2 \cos^2\theta.$$

Graphing these relations between wave speed and $\sin^2\theta$ for the generic cases $0 < v_A^2 < v_s^2$ and $v_A^2 > v_s^2 > 0$ characterizes the general relationship among MHD plasma wave speeds (see Figure 1).

Recall the eigenvector system

$$\begin{bmatrix} c & -1 & 0 & 0 & 0 \\ 0 & g & 0 & -c & 0 \\ 0 & 0 & c & 0 & B_1 \\ 0 & 0 & 0 & c^2 B_2 & g(B_1 B_1^* - c^2) \\ 0 & 0 & 0 & gg^* - c^2 & gB_2^* \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0,$$

that is,

$$\begin{bmatrix} c & -1 & 0 & 0 & 0 \\ 0 & \gamma p & 0 & -c & 0 \\ 0 & 0 & c & 0 & B_1 \\ 0 & 0 & 0 & c^2 B_2 & \gamma p(c_a^2 - c^2) \\ 0 & 0 & 0 & v_s^2 - c^2 & v_s^2 B_2 \end{bmatrix}_0 \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}' = 0.$$

One's choice among the bottom two equations gives two possible ways to express the right eigenvectors:

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}'_{\text{right}} \propto \begin{pmatrix} cB_2^* \\ c^2 B_2^* \\ B_1(gg^* - c^2) \\ cgB_2^* \\ c(c^2 - gg^*) \end{pmatrix} \propto \begin{pmatrix} c^2 - B_1 B_1^* \\ c(c^2 - B_1 B_1^*) \\ -cB_1 B_2 \\ gg^*(c^2 - B_1 B_1^*) \\ c^2 B_2 \end{pmatrix}$$

That is,

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}'_{\text{right}} \propto \begin{pmatrix} c^2 - c_A^2 \\ c(c^2 - c_A^2) \\ -cB_1 B_2 \\ v_s^2(c^2 - c_A^2) \\ c^2 B_2 \end{pmatrix}.$$

For the left eigenvectors, transposing the system matrix and using c to kill 1 in the first column (which assumes $c \neq 0$) shows that to get the left eigenvalues we just zero out the density perturbation and swap the starred and unstarred variables:

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ p \\ B_2 \end{pmatrix}'_{\text{left}} \propto \begin{pmatrix} 0 \\ c^2 B_2 \\ B_1^*(gg^* - c^2) \\ cgB_2 \\ c(c^2 - gg^*) \end{pmatrix} \propto \begin{pmatrix} 0 \\ c(c^2 - B_1 B_1^*) \\ -cB_1^* B_2^* \\ gg^*(c^2 - B_1 B_1^*) \\ c^2 B_2^* \end{pmatrix}.$$

1.3 Case $B_1 = 0$

...

1.4 Case $B_2 = 0$

...

2 Alfvén waves

In the case of Alfvén waves, the fact that the wave speed is independent of the perturbation variables suggests that we should seek a finite-amplitude solution of the nonlinearized MHD equations.

Recall 1-dimensional MHD in quasilinear form:

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}_{,t} + \begin{bmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & g^* & B_2^* & B_3^* \\ 0 & 0 & u_1 & 0 & 0 & -B_1^* & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & -B_1^* \\ 0 & g & 0 & 0 & u_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & u_1 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & u_1 \end{bmatrix} \cdot \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \\ B_2 \\ B_3 \end{pmatrix}_{,x} = 0.$$

Without linearizing, we seek a solution for which ρ and p are constant. The evolution equation for ρ then implies that u_1 is constant in space, and hence constant in time by global momentum conservation. The evolution equation for u_1 in turn simplifies to $\partial_t u_1 + B_2^* \partial_x B_2 + B_3^* \partial_x B_3 = 0$, i.e., $0 = \partial_x (B_2^2 + B_3^2)$, which says that $B_2^2 + B_3^2$ must be constant in space. Transforming into the frame of reference of the fluid, we may say without loss of generality that $u_1 = 0$.

So our system simplifies to

$$\begin{pmatrix} u_2 \\ B_2 \end{pmatrix}_{,t} + \begin{bmatrix} 0 & -B_1^* \\ -B_1 & 0 \end{bmatrix} \cdot \begin{pmatrix} u_2 \\ B_2 \end{pmatrix}_{,x} = 0$$

and

$$\begin{pmatrix} u_3 \\ B_3 \end{pmatrix}_{,t} + \begin{bmatrix} 0 & -B_1^* \\ -B_1 & 0 \end{bmatrix} \cdot \begin{pmatrix} u_3 \\ B_3 \end{pmatrix}_{,x} = 0.$$

This is just a pair of wave equations with speeds $c = \pm\sqrt{B_1 B_1^*} = \pm\frac{B_1}{\sqrt{\mu_0 \rho}}$ and corresponding right- and left-traveling waves

$$\begin{pmatrix} u_i \\ B_i \end{pmatrix} = f_i^{\pm}(x - ct) \begin{pmatrix} \mp 1 \\ \sqrt{\mu_0 \rho} \end{pmatrix},$$

where the $f_i^{\pm}$ need to satisfy the requirement that $(f_2^+)^2 + (f_3^+)^2$ and $(f_2^-)^2 + (f_3^-)^2$ are constant.

For a ready such pair of functions, choose $f_2(x) = u_0 \cos(kx)$ and $f_3(x) = u_0 \sin(kx)$. This gives the solution

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \mp u_0 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

and

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = B_0 \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix},$$

where

$$u_0 = \frac{B_0}{\sqrt{\mu_0 \rho}}, \quad \theta := kx - \omega t, \quad \text{and} \quad \frac{\omega}{k} = c = \frac{\pm B_1}{\sqrt{\mu_0 \rho}}.$$

For this solution the components of **u** and **B** perpendicular to the x-axis are rotationally polarized and aligned or anti-aligned depending on whether the wave is propagating in the negative or positive direction.