Linearized plasma

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   - Waves and eigenstructure
2. MHD
   - waves
   - instabilities
3. Two-fluid
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Linear stability

Linearization yields analytical solutions near an equilibrium which we can test for stability.

Consider a dynamical system:

$$d_t X = F(X).$$

Suppose that X_0 is an equilibrium:

$$F(X_0) = 0.$$

Taylor expansion gives:

$$d_t X' \approx F_{,X} \cdot X',$$

where $X' = X - X_0$ is the perturbation from equilibrium and $F_{,X}$ is the matrix of partial derivatives $\frac{\partial F^j}{\partial X^k}$ and where Taylor says that the approximation is exact if $F_{,X}$ is evaluated at $X_0 + \theta X'$ for the right value $0 \leq \theta \leq 1$.

The linearized system is

$$d_t X' = \mathbb{A} \cdot X',$$

where the matrix of constant coefficients is evaluated at the equilibrium state:

$$\mathbb{A} := F_{,X}(X_0).$$

If $\mathbb{A}$ has an eigenvalue with *positive real part* then X_0 is a (linearly) **unstable** equilibrium; else X_0 is **linearly stable**.

We use the comma-subscript notation for partial derivatives in these slides.

Consider a hyperbolic balance law

$$\text{(CON)} \quad Q_{,t} + F^j(Q)_{,x^j} = S(Q),$$

where there is an implicit sum over the spatial index j.

Remark: These slides concern one-dimensional plane waves. Therefore, you can choose to erase all repeated indices (so read F^j as F and x^j as x) and assume that $\partial_y = 0 = \partial_z$.

Background states:

- We define a **background state** (or ***an*** *equilibrium*) to be a time-independent solution $Q_0(\mathbf{x})$.
- A ***uniform*** *background state* is independent of space: $S(Q_0) \equiv 0$.

To linearize system (CON), we first put it in **quasilinear form** using the chain rule:

$$\text{(QL)} \quad Q_{,t} + F^j_{,Q} \cdot Q_{,x^j} = S;$$

To complete the linearization, we Taylor-expand the source term:

$$S \approx S(Q_0) + S_{,Q} \cdot Q',$$

where $Q' := Q - Q_0$, and freeze the coefficients by evaluating them at Q_0:

$$\mathbb{A}^i := F^j_{,Q}(Q_0) \quad \text{and}$$
$$\mathbb{S} := S_{,Q}(Q_0).$$

This gives the constant-coefficient linearized system

$$\text{(LIN)} \quad Q'_{,t} + \mathbb{A}^i \cdot Q'_{,x^j} = \mathbb{S} \cdot Q'.$$

Wave and stability analysis

Linearization is used to study fluid models in three different ways:

1. To study linear waves we assume a uniform background state that maximizes entropy.

   In the two-fluid case, maximum entropy means zero drift ($\mathbf{u}_\mathrm{i} = \mathbf{u}_\mathrm{e}$), equilibrated temperatures ($T_\mathrm{i} = T_\mathrm{e}$), and charge neutrality ($n_\mathrm{i} = n_\mathrm{e}$). Any MHD uniform background state maximizes entropy, since MHD maximizes local entropy. We assume without loss of generality that the background fluid velocity is zero.

2. For *two-fluid linear stability analysis*, we assume a *uniform* background state that does not maximize entropy (e.g. assigning different velocities to the two fluids). See e.g. Nicholson Chapter 7 (section 7.13).

   Remark: the two-fluid model is relevant to small scales, and is used to study microscopic instabilities where it is reasonable to assume a uniform background state.

3. For *MHD linear stability analysis*, one assumes a *nonuniform* background state.

   Remark: MHD applies to large scales, and is used to study macroscopic stability of nonuniform large-scale equilibrium configurations — see Nicholson Chapter 8.

This document is concerned with uniform background states and focuses on linear waves (case 1).

Primitive variables (versus conserved)

Conserved state variables are densities of conserved quantities and are used to write the equations in balance form.

For MHD, conserved variables are

$$Q_{(\mathrm{MHD})} := (\rho, \rho\mathbf{u}, \mathcal{E}_{\mathrm{MHD}}, \mathbf{B}),$$

where $\mathcal{E}_{\mathrm{MHD}} := \mathcal{E} + \frac{1}{2\mu_0}\|\mathbf{B}\|^2$ is the sum of gas-dynamic energy and magnetic field energy.

For two-fluid plasma, conserved variables are

$$Q_{(2\mathrm{fluid})} := (\rho_{\mathrm{i}}, \rho_{\mathrm{i}}\mathbf{u}_{\mathrm{i}}, \mathcal{E}_{\mathrm{i}}, \rho_{\mathrm{e}}, \rho_{\mathrm{e}}\mathbf{u}_{\mathrm{e}}, \mathcal{E}_{\mathrm{e}}, \mathbf{B}, \mathbf{E}).$$

Primitive state variables are simpler quantities and are used to write the equations in a simpler system.

For MHD, primitive variables are

$$P_{(\mathrm{MHD})} := (\rho, \mathbf{u}, p, \mathbf{B}),$$

where p does *not* include the magnetic pressure $p_B := \frac{|\mathbf{B}|^2}{2\mu_0}$.

For two-fluid plasma, primitive variables are

$$Q_{(2\mathrm{fluid})} := (\rho_{\mathrm{i}}, \mathbf{u}_{\mathrm{i}}, p_{\mathrm{i}}, \rho_{\mathrm{e}}, \mathbf{u}_{\mathrm{e}}, p_{\mathrm{e}}, \mathbf{B}, \mathbf{E}).$$

Pressure is proportional to thermal energy density:

$$p = \frac{2}{3}\left(\mathcal{E} - \frac{1}{2}\rho|\mathbf{u}|^2\right).$$

Linearization in primitive variables (versus conserved)

- To convert the hyperbolic balance law

$$Q_{,t} + F^j_{,x^j} = S$$

to primitive variables, multiply by $P_{,Q}$ and use the chain rule to get:

$$P_{,t} + P_{,Q} \cdot F^j_{,x^j} = P_{,Q} \cdot S =: \widetilde{S}$$

- The chain rule yields the **quasilinear system in the variables** P:

$$P_{,t} + \mathbb{A}^j_P \cdot P_{,x^j} = \widetilde{S},$$

where $\mathbb{A}^j_P = P_{,Q} \cdot F^j_{,P}$.

- For a ***uniform background state***, the fully linearized primitive system is

$$\boxed{P'_{,t} + \mathbb{A}^j_P \cdot P'_{,x^j} = \mathbb{S}_P \cdot P',}$$

where $\mathbb{S}_P := \widetilde{S}_{,P}$ and $\mathbb{A}^j_P$ are evaluated at the background state P_0 and $P' := P - P_0$.

- **Conjugate eigenstructure.** The coefficient matrices of the primitive system are conjugates of the coefficient matrices of the conserved system and therefore have the same eigenvalues and equivalent eigenvectors:

$$\begin{aligned}
\mathbb{A}^j_P &= P_{,Q} \cdot F^j_{,P} \\
&= P_{,Q} \cdot F^j_{,Q} \cdot Q_{,P}. \\
&= P_{,Q} \cdot \mathbb{A}^j_Q \cdot Q_{,P}, \quad \text{and} \\
\mathbb{S}_P :&= \widetilde{S}_{,P} \\
&= P_{,Q} \cdot S_{,Q} \cdot Q_{,P}. \\
&= P_{,Q} \cdot \mathbb{S}_Q \cdot Q_{,P}.
\end{aligned}$$

Note: $P_{,Q}$ and $Q_{,P}$ are inverse matrices, since by the chain rule $P_{,Q} \cdot Q_{,P} = P_{,P} = \mathbb{I}$.

Work in primitive variables for easy linearization.

- To put evolution equations for P in quasilinear form, apply differentiation rules until no spatial derivatives remain except derivatives of components of P.
- Remark 1: We will see that linear waves are revealed by the eigenstructure e.g. of $\mathbb{A}^j_P$. Quasilinear form is usually simpler in primitive variables, and it is usually easier to find the eigenstructure for $\mathbb{A}^j_P$ than for $\mathbb{A}^i_Q$.
- Remark 2: Since

$$\begin{aligned} \mathbb{A}^i_Q &= P_{,Q} \cdot \mathbb{A}^j_P \cdot Q_{,P} \quad \text{and} \\ \mathbb{S}_Q &= P_{,Q} \cdot \mathbb{S}_P \cdot Q_{,P} \end{aligned}$$

  it is possible (and sometimes convenient) to find the eigenstructure for the conserved variables without ever having to express the equations in conserved variables!

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Dispersion relation for a hyperbolic balance law

- As we have seen, linearization in any set of state variables is equivalent to any other set, whether conserved or primitive.
- Henceforth let U denote an arbitrary set of state variables.
- Consider a linearized system with constant coefficients:

$$\boxed{U_{,t} + \mathbb{A}^i \cdot U_{,x^i} = \mathbb{S} \cdot U,}$$

  where constant-coefficient matrices are used for $\mathbb{A}$ and $\mathbb{S}$.
- Seek eigensolutions of the form

$$U = U_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}.$$

  Then:

$$U_{,t} = -i\omega U.$$
$$U_{,\mathbf{x}} = i\mathbf{k}U.$$

- Get the **dispersion relation**

$$\boxed{\left(k\widehat{\mathbb{A}} + i\mathbb{S}\right) \cdot U_0 = \omega U_0,}$$

  where $\widehat{\mathbb{A}} := \widehat{\mathbf{k}} \cdot \mathbb{A}$.
- This is an ***eigenvalue problem***.
- Without loss of generality, we will choose $\widehat{\mathbf{k}}$ to be aligned with the x axis.
- For each choice of $\mathbf{k}$ (and for each choice of U_0) there exists a set of up to N eigenvalues ω, where N is the number of variables in U.
- For dispersion relations, the background state is characterized by the number density n_0, the equilibrium temperature T_0, and the magnetic field vector $\mathbf{B}_0$.

Hyperbolic conservation laws yield nondispersive shocks (Ideal MHD)

For hyperbolic conservation laws such as MHD, the source term $\mathbb{S}$ is zero, so the dispersion relation is simply $k\widehat{\mathbb{A}} \cdot U_0 = \omega U_0$. That is:

$$\text{(DIS)} \quad \boxed{\widehat{\mathbb{A}} \cdot U_0 = \left(\frac{\omega}{k}\right) U_0};$$

the eigenvalues $\lambda = \frac{\omega}{k}$ represent wave speeds and are independent of k.

Discontinuous shocks can form:

- **Nondispersion.** (DIS) says that for hyperbolic conservation laws, *wave speeds are determined entirely by the state and are independent of the wave frequency*. That is, *hyperbolic conservation laws are nondispersive.*
- **Shock formation.** A major consequence of nondispersion is that discontinuous shocks can form; wave speeds are determined by state value alone and are not affected by the development of steep gradients, so nonlinear waves simply steepen until they become shocks. The presence of discontinuous shocks in ideal MHD makes it fundamentally different from the two-fluid model.

Shocks and finite-amplitude waves (Ideal MHD)

Assuming 1D and leaving in conservation form

$$\partial_t Q + F_x = 0$$

reveals the **jump condition** that MHD *shock waves* must satisfy:

$$d_t \int_{x_0}^{x_1} Q + [F]_{x_0}^{x_1} = 0, \quad \text{so}$$

$$\dot{s}\,[Q]_{s-\epsilon}^{s+\epsilon} = [F]_{s-\epsilon}^{s+\epsilon} \text{ as } \epsilon \to 0,$$

where $s(t)$ is shock position. That is, the jump in flux across a shock equals shock speed times the jump in the conserved state variables. This says that, in the frame of reference of the shock, conserved material enters and emerges from the shock at the same rate. This is called the *Rankine-Hugoniot* ***jump condition***.

Remarks

- The jump condition holds for conserved variables only, not primitive variables.
- For MHD, assuming 1D without linearizing reveals finite-amplitude linear (Alfven) and *nonlinear* waves. Nonlinear waves generically develop shocks.
- Shocks locally appear 1D and so are essentially a 1D phenomenon.
- In fact, 1D waves characterize fluid behavior.
- What defines a wave? *A wave is a propagating modulation in value*.

- Recall the general form of the two-fluid Euler-Maxwell dispersion relation:

$$\boxed{\left(k\widehat{\mathbb{A}} + i\mathbb{S}\right) \cdot U_0 = \omega U_0.}$$

- In the two-fluid case, given a background state, there is a different set of wave speeds for every choice of wave number k.
- The two-fluid dispersion relation can be characterized in terms of high-frequency and low-frequency limits.

Limiting cases

- **$k \to \infty$, $\omega \to \infty$: hyperbolic conservation law ($\mathbb{S} = 0$)**

  As $k \to \infty$, the source term vanishes and the dispersion relation is

  $$\widehat{\mathbb{A}} \cdot U_0 = \left(\tfrac{\omega}{k}\right) U_0;$$

  for the two-fluid Maxwell system, this represents the limit where there is no source term and the system decouples into gas dynamics for each species and propagation of light in a vacuum. The eigenvalues $\lambda = \frac{\omega}{k}$ become independent of k and approach the ion and electron *sound wave* speeds and the *speed of light*.

- **$k \to 0$, $\omega > 0$: source term system ($\mathbb{A} = 0$)**

  As $k \to 0$, the dispersion relation is the eigenstructure problem for the source term ODE:

  $$\mathbb{S} \cdot U_0 = -i\omega U_0;$$

  for the two-fluid model, $\mathbb{S}$ has imaginary eigenvalues, resulting in undamped oscillations called *Langmuir waves*.

Waves: eigenstructure of the dispersion relation (diffusive)

Dispersion for a parabolic balance law

- Given a linearized system with constant coefficients:

$$\boxed{U_{,t} + \mathbb{A}^i \cdot U_{,x^j} = \mathbb{S} \cdot U + \mathbb{G}^{ik} U_{,x^j x^k},}$$

  where constant-coefficient matrices are used for $\mathbb{A}$, $\mathbb{S}$, and $\mathbb{G}$,
- seek eigensolutions of the form

$$U = U_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}.$$

- Get the dispersion relation

$$\boxed{\left(k\widehat{\mathbb{A}} + i\mathbb{S} - ik^2\widehat{\mathbb{G}}\right) \cdot U_0 = \omega U_0,}$$

  where $\widehat{\mathbb{A}} := \widehat{\mathbf{k}} \cdot \mathbb{A}$ and $\widehat{\mathbb{G}} := \widehat{\mathbf{k}}\widehat{\mathbf{k}} : \mathbb{G}$.
- Recall: WLOG we can choose $\widehat{\mathbf{k}} = \widehat{x}$. So to analyze waves we can assume one-dimensional systems:

$$\boxed{\partial_t U + \widehat{\mathbb{A}} \cdot U_x = \mathbb{S} \cdot U + \widehat{\mathbb{G}} \cdot U_{xx}.}$$

Remarks

- Diffusive terms arise from viscosity, heat flux, and resistivity:

$$\begin{aligned} \mathbb{P}^\circ &\approx -2\boldsymbol{\mu} : \nabla \mathbf{u}^\circ, \\ \mathbf{q} &\approx -\mathbf{k} \cdot \nabla T, \\ \mathbf{E} &\approx \mathbf{B} \times \mathbf{u} + \boldsymbol{\eta} \cdot \mathbf{J}. \end{aligned}$$

  Viscosity and heat flux arise when intraspecies collisions are not instantaneous: $\tau_{\mathrm{ss}} \neq 0$. Resistivity arises when interspecies collisions occur: $\tau_{\mathrm{ie}}^{-1} \neq 0$.
- Diffusion damps high-frequency waves. For $k \to \infty$,

$$\widehat{\mathbb{G}} \cdot U_0 \approx \lambda U_0, \text{ where } \lambda = \frac{i\omega}{k^2}.$$

  So $U \approx U_0 e^{i\mathbf{k} \cdot \mathbf{x}} e^{-\lambda k^2 t}$, which decays if $\widehat{\mathbb{G}}$ has positive eigenvalues; if $\widehat{\mathbb{G}}$ has a negative eigenvalue then the closure is *antidiffusive*, and the model is *ill-posed*.

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MHD waves

Linearized MHD waves are perturbations on a background state with given density, temperature, and magnetic field.

Given a choice of background state, wave speeds and eigenstructure is determined by the *direction* $\widehat{\mathbf{k}}$ of the magnetic field. After shifting and rescaling, the solution is entirely determined by two parameters:

1. $\widehat{\mathbf{k}} \cdot \widehat{\mathbf{B}} = \cos\theta$ (θ is the angle between the magnetic field and wave direction) and
2. $\beta := \frac{p_0}{p_\mathrm{B}} = \left(\frac{v_{t,s}}{v_{A,s}}\right)^2$ (the ratio of pressure to magnetic pressure $p_\mathrm{B} := \frac{|\mathbf{B}|^2}{2\mu_0}$).

The most important angles are perpendicular and parallel to the magnetic field. In these cases, MHD has three fundamental waves:

1. **Alfvén waves** are transverse oscillations that propagate parallel to the magnetic field at speed $v_\mathrm{A} = \sqrt{2p_\mathrm{B}/\rho}$.
2. Pure **sound waves** are compressive waves that propagate parallel to the magnetic field at speed $v_\mathrm{s} = \sqrt{\frac{\gamma p}{\rho}}$.
3. **Fast magnetosonic waves** are compressive waves that propagate perpendicular to the magnetic field at speed $v_\mathrm{f} = \sqrt{v_\mathrm{s}^2 + v_\mathrm{A}^2}$.

For a general direction $\widehat{\mathbf{k}}$, there are three wave speeds:

1. Oblique **Alfvén waves** with speed $c_\mathrm{A} = v_\mathrm{A}|\cos\theta|$,
2. *Slow magnetosonic waves* with speed c_s satisfying $$0 \le c_\mathrm{s}^2 = \frac{1}{2}\left[v_\mathrm{f}^2 - \sqrt{v_\mathrm{f}^2 - 4v_\mathrm{s}^2 c_\mathrm{A}^2}\right] \le \min\{v_\mathrm{A}^2, v_\mathrm{s}^2\}$$
3. *Fast magnetosonic waves* with speed c_f satisfying $$v_\mathrm{f} \ge c_\mathrm{f}^2 = \frac{1}{2}\left[v_\mathrm{f}^2 + \sqrt{v_\mathrm{f}^2 - 4v_\mathrm{s}^2 c_\mathrm{A}^2}\right] \ge \max\{v_\mathrm{A}^2, v_\mathrm{s}^2\}$$

See my note, "Waves in MHD", for a full derivation.

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MHD instabilities

MHD is used to study the stability of macroscopic plasma configurations. Controling these instabilities is critical to the project of fusion energy via magnetic confinement.

A linear analysis of an instability (e.g. sausage, kink/firehose, ballooning) proceeds by linearizing about a steady-state spatially dependent solution.

Stability can be proved by showing that all perturbations entail an increase in free energy.

See Nicholson §8.3.

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Quasi-linearization

The **two-fluid Euler Maxwell** system is the two-fluid Maxwell system without any diffusive or collisional terms.

Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\epsilon_0,$$
$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0.$$

Evolution equations:

$$\partial_t \rho_s + \nabla \cdot (\mathbf{u}_s \rho_s) = 0,$$
$$\rho_s d_t^s \mathbf{u}_s + \nabla p_s = \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B}$$
$$\rho_s d_t^s e_s + p_s \nabla \cdot \mathbf{u}_s = 0$$

Written in quasilinear form in the primitive variables

$$P := (\rho_i, \mathbf{u}_i, p_i, \rho_e, \mathbf{u}_e, p_e, \mathbf{B}, \mathbf{E}),$$

the 2-fluid Euler-Maxwell system is:

Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\left(\frac{e}{m_e}\rho_e - \frac{e}{m_e}\rho_i\right),$$
$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = \frac{1}{\epsilon_0}\left(\frac{e}{m_e}\rho_e \mathbf{u}_e - \frac{e}{m_e}\rho_i \mathbf{u}_i\right),$$

Evolution equations:

$$\partial_t \rho_s + \mathbf{u}_s \cdot \nabla \rho_s + \rho_s \nabla \cdot \mathbf{u}_s = 0,$$
$$\partial_t \mathbf{u}_s + \mathbf{u}_s \cdot \nabla \mathbf{u}_s + \frac{1}{\rho} \nabla p_s = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}),$$
$$\partial_t p_s + \mathbf{u}_s \cdot \nabla p_s + \gamma p_s \nabla \cdot \mathbf{u}_s = 0,$$

where $\gamma := \frac{5}{3}$ is the adiabatic index.

To identify the quasilinear matrix coefficients, we line up derivatives.

For the ion gas dynamics equations, we have:

$$
\begin{aligned}
0 &= \partial_t \rho_\mathrm{i} + \mathbf{u}_\mathrm{i}\cdot\nabla\rho_\mathrm{i} + \rho_\mathrm{i}\nabla\cdot\mathbf{u}_\mathrm{i},\\
\mathbf{s}^\mathrm{i} &= \partial_t \mathbf{u}_\mathrm{i} + \mathbf{u}_\mathrm{i}\cdot\nabla\mathbf{u}_\mathrm{i} + \frac{1}{\rho_\mathrm{i}}\nabla p_\mathrm{i},\\
0 &= \partial_t p_\mathrm{i} + \gamma p_\mathrm{i}\nabla\cdot\mathbf{u}_\mathrm{i} + \mathbf{u}_\mathrm{i}\cdot\nabla p_\mathrm{i},
\end{aligned}
$$

where the source term is defined by:

$$
\mathbf{s}^\mathrm{i} := \frac{\mathbf{q}_\mathrm{i}}{m_\mathrm{i}}(\mathbf{E} + \mathbf{u}_\mathrm{i}\times\mathbf{B}).
$$

For a one-dimensional problem, $0 = \partial_y = \partial_z$, so $\nabla = \widehat{x}\partial_x$ and this becomes:

$$
\begin{aligned}
0 &= \partial_t \rho_\mathrm{i} + u_\mathrm{i}^1\partial_x\rho_\mathrm{i} + \rho_\mathrm{i}\partial_x u_\mathrm{i}^1,\\
\mathbf{s}^\mathrm{i} &= \partial_t \mathbf{u}_\mathrm{i} + u_\mathrm{i}^1\partial_x\mathbf{u}_\mathrm{i} + \frac{1}{\rho_\mathrm{i}}\widehat{x}\partial_x p_\mathrm{i},\\
0 &= \partial_t p_\mathrm{i} + \gamma p_\mathrm{i}\partial_x u_\mathrm{i}^1 + u_\mathrm{i}^1\partial_x p_\mathrm{i},
\end{aligned}
$$

In matrix form:

$$
\begin{pmatrix}\rho_\mathrm{i}\\ u_\mathrm{i}^1\\ u_\mathrm{i}^2\\ u_\mathrm{i}^3\\ p_\mathrm{i}\end{pmatrix}_t
+ \underbrace{\begin{bmatrix}
u_\mathrm{i}^1 & \rho_\mathrm{i} & 0 & 0 & 0\\
0 & u_\mathrm{i}^1 & 0 & 0 & \frac{1}{\rho_\mathrm{i}}\\
0 & 0 & u_\mathrm{i}^1 & 0 & 0\\
0 & 0 & 0 & u_\mathrm{i}^1 & 0\\
0 & \gamma p_\mathrm{i} & 0 & 0 & u_\mathrm{i}^1
\end{bmatrix}}_{\text{Calling } \mathbb{A}^\mathrm{i}}
\cdot
\begin{pmatrix}\rho_\mathrm{i}\\ u_\mathrm{i}^1\\ u_\mathrm{i}^2\\ u_\mathrm{i}^3\\ p_\mathrm{i}\end{pmatrix}_x
=
\begin{pmatrix}0\\ \mathbf{s}^{\mathrm{i}1}\\ \mathbf{s}^{\mathrm{i}2}\\ \mathbf{s}^{\mathrm{i}3}\\ 0\end{pmatrix}
=
\underbrace{\begin{pmatrix}
0\\
\frac{\mathbf{q}_\mathrm{i}}{m_\mathrm{i}}(E^1 + u_\mathrm{i}^2 B^3 - u_\mathrm{i}^3 B^2)\\
\frac{\mathbf{q}_\mathrm{i}}{m_\mathrm{i}}(E^2 + u_\mathrm{i}^3 B^1 - u_\mathrm{i}^1 B^3)\\
\frac{\mathbf{q}_\mathrm{i}}{m_\mathrm{i}}(E^3 + u_\mathrm{i}^1 B^2 - u_\mathrm{i}^2 B^1)\\
0
\end{pmatrix}}_{\text{Calling } S^\mathrm{i}}.
$$

That is, formally:

$$
\partial_t P^\mathrm{i} + \mathbb{A}^\mathrm{i}\cdot P_x^\mathrm{i} = S^\mathrm{i}
$$

Maxwell in matrix form

Maxwell's equations are already linear:

$$\partial_t \mathbf{B} + c_1 \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c_2 \nabla \times \mathbf{B} = \mathbf{s}^{\mathbf{E}},$$

where for SI units $c_1 = 1$ and $c_2 = c^2$ and

$$\mathbf{s}^{\mathbf{E}} := \frac{e}{\epsilon_0}\left(\frac{\rho_\mathrm{e}}{m_\mathrm{e}}\mathbf{u}_\mathrm{e} - \frac{\rho_\mathrm{i}}{m_\mathrm{i}}\mathbf{u}_\mathrm{i}\right).$$

If $0 = \partial_x = \partial_y$, then this is of the form

$$\partial_t P^\mathrm{M} + \mathbb{A}^\mathrm{M} \cdot P^\mathrm{M}_x = S^\mathrm{M}.$$

This one-dimensional system is three subsystems that are independent if gas-dynamic quantities are prescribed:

$$\begin{pmatrix} B^1 \\ E^1 \end{pmatrix}_{,t} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} B^1 \\ E^1 \end{pmatrix}_{,x} = \begin{pmatrix} 0 \\ \mathbf{s}^{\mathbf{E}1} \end{pmatrix},$$
$$\begin{pmatrix} B^3 \\ E^2 \end{pmatrix}_{,t} + \begin{bmatrix} 0 & c_1 \\ -c_2 & 0 \end{bmatrix} \cdot \begin{pmatrix} B^3 \\ E^2 \end{pmatrix}_{,x} = \begin{pmatrix} 0 \\ \mathbf{s}^{\mathbf{E}2} \end{pmatrix},$$
$$\begin{pmatrix} B^2 \\ E^3 \end{pmatrix}_{,t} + \begin{bmatrix} 0 & -c_1 \\ c_2 & 0 \end{bmatrix} \cdot \begin{pmatrix} B^2 \\ E^3 \end{pmatrix}_{,x} = \begin{pmatrix} 0 \\ \mathbf{s}^{\mathbf{E}3} \end{pmatrix}.$$

Written out in full:

$$\partial_t \begin{pmatrix} B^1 \\ B^2 \\ B^3 \\ E^1 \\ E^2 \\ E^3 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_1 \\ 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_2 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Calling } \mathbb{A}^\mathrm{M}} \cdot \begin{bmatrix} B^1 \\ B^2 \\ B^3 \\ E^1 \\ E^2 \\ E^3 \end{bmatrix}_{,x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{s}^{\mathbf{E}1} \\ \mathbf{s}^{\mathbf{E}2} \\ \mathbf{s}^{\mathbf{E}3} \end{pmatrix} = \frac{e}{\epsilon_0}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\rho_\mathrm{e}}{m_\mathrm{e}} u_\mathrm{e}^1 - \frac{\rho_\mathrm{i}}{m_\mathrm{i}} u_\mathrm{i}^1 \\ \frac{\rho_\mathrm{e}}{m_\mathrm{e}} u_\mathrm{e}^2 - \frac{\rho_\mathrm{i}}{m_\mathrm{i}} u_\mathrm{i}^2 \\ \frac{\rho_\mathrm{e}}{m_\mathrm{e}} u_\mathrm{e}^3 - \frac{\rho_\mathrm{i}}{m_\mathrm{i}} u_\mathrm{i}^3 \end{pmatrix},$$

Composite quasi-linear system

The composite quasilinear system is thus:

$$\begin{bmatrix} P^{\mathrm{i}} \\ P^{\mathrm{e}} \\ P^{\mathrm{M}} \end{bmatrix}_{,t} + \begin{bmatrix} \mathbb{A}^{\mathrm{i}} & 0 & 0 \\ 0 & \mathbb{A}^{\mathrm{e}} & 0 \\ 0 & 0 & \mathbb{A}^{\mathrm{M}} \end{bmatrix}_{,t} \cdot \begin{bmatrix} P^{\mathrm{i}} \\ P^{\mathrm{e}} \\ P^{\mathrm{M}} \end{bmatrix}_{,x} = \begin{bmatrix} S^{\mathrm{i}} \\ S^{\mathrm{e}} \\ S^{\mathrm{M}} \end{bmatrix}.$$

Linearization about a uniform background state P_0 approximates the source term by $\widetilde{S} \approx \widetilde{S}_{,P} \cdot P'$:

$$\begin{bmatrix} S^{\mathrm{i}} \\ S^{\mathrm{e}} \\ S^{\mathrm{M}} \end{bmatrix} \approx \begin{bmatrix} S^{\mathrm{i}}_{,P^{\mathrm{i}}} & 0 & S^{\mathrm{i}}_{,\mathrm{M}} \\ 0 & S^{\mathrm{e}}_{,P^{\mathrm{e}}} & S^{\mathrm{e}}_{,P^{\mathrm{M}}} \\ S^{\mathrm{M}}_{,P^{\mathrm{i}}} & S^{\mathrm{M}}_{,P^{\mathrm{e}}} & 0 \end{bmatrix} \cdot \begin{bmatrix} P^{\mathrm{i}} \\ P^{\mathrm{e}} \\ P^{\mathrm{M}} \end{bmatrix}',$$

where $P' := P - P_0$. More fully, using "$\cdot$" for 0:

$$\left[\begin{array}{c} \cdot \\ \mathbf{s}^{\mathrm{i}} \\ \cdot \\ \hline \cdot \\ \mathbf{s}^{\mathrm{e}} \\ \cdot \\ \hline \cdot \\ \mathbf{s}^{\mathrm{E}} \end{array}\right] \approx \left[\begin{array}{ccc|ccc|cc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathbf{s}^{\mathrm{i}}_{,P^{\mathrm{i}}} & \cdot & \cdot & \cdot & \cdot & \mathbf{s}^{\mathrm{i}}_{,\mathbf{B}} & \mathbf{s}^{\mathrm{i}}_{,\mathbf{E}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathbf{s}^{\mathrm{e}}_{,P^{\mathrm{e}}} & \cdot & \mathbf{s}^{\mathrm{e}}_{,\mathbf{B}} & \mathbf{s}^{\mathrm{e}}_{,\mathbf{E}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{s}^{\mathbf{E}}_{,\rho_{\mathrm{i}}} & \mathbf{s}^{\mathbf{E}}_{,\mathbf{u}_{\mathrm{i}}} & \cdot & \mathbf{s}^{\mathbf{E}}_{,\rho_{\mathrm{e}}} & \mathbf{s}^{\mathbf{E}}_{,\mathbf{u}_{\mathrm{e}}} & \cdot & \cdot & \cdot \end{array}\right] \cdot \left[\begin{array}{c} \rho_{\mathrm{i}} \\ \mathbf{u}_{\mathrm{i}} \\ p_{\mathrm{i}} \\ \hline \rho_{\mathrm{e}} \\ \mathbf{u}_{\mathrm{e}} \\ p_{\mathrm{e}} \\ \hline \mathbf{B} \\ \mathbf{E} \end{array}\right]'$$

Definitions:

$$P := \begin{bmatrix} P^{\mathrm{i}} \\ P^{\mathrm{e}} \\ P^{\mathrm{M}} \end{bmatrix} \quad P^{\mathrm{i}} := \begin{bmatrix} \rho_{\mathrm{i}} \\ \mathbf{u}_{\mathrm{i}} \\ p_{\mathrm{i}} \end{bmatrix} \quad P^{\mathrm{e}} := \begin{bmatrix} \rho_{\mathrm{e}} \\ \mathbf{u}_{\mathrm{e}} \\ p_{\mathrm{e}} \end{bmatrix} \quad P^{\mathrm{M}} := \begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix}$$

$$S^{\mathrm{i}} := \begin{bmatrix} 0 \\ \mathbf{s}^{\mathrm{i}} \\ 0 \end{bmatrix} \quad S^{\mathrm{e}} := \begin{bmatrix} 0 \\ \mathbf{s}^{\mathrm{e}} \\ 0 \end{bmatrix} \quad S^{\mathrm{M}} := \begin{bmatrix} 0 \\ \mathbf{s}^{\mathbf{E}} \end{bmatrix}$$

(Definitions:)

$$\mathbf{s}^{\mathbf{E}} := \frac{1}{\epsilon_0} \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \mathbf{u}_\alpha$$

$$\mathbf{s}^{\mathrm{i}} := \frac{q_{\mathrm{i}}}{m_{\mathrm{i}}} (\mathbf{E} + \mathbf{u}_{\mathrm{i}} \times \mathbf{B}).$$

Derivatives:

$$\mathbf{s}^{\mathrm{i}}_{,p^{\mathrm{i}}} := \frac{q_{\mathrm{i}}}{m_{\mathrm{i}}} \mathbb{I} \times \mathbf{B},$$

$$\mathbf{s}^{\mathbf{E}}_{,\rho_{\mathrm{i}}} := \frac{1}{\epsilon_0} \frac{q_{\mathrm{i}}}{m_{\mathrm{i}}} \mathbf{u}_{\mathrm{i}},$$

$$\mathbf{s}^{\mathbf{E}}_{,\mathbf{u}_{\mathrm{i}}} := \frac{1}{\epsilon_0} \frac{q_{\mathrm{i}}}{m_{\mathrm{i}}} \rho_{\mathrm{i}} \mathbb{I},$$

$$\mathbf{s}^{\mathrm{i}}_{,\mathbf{E}} := \frac{q_{\mathrm{i}}}{m_{\mathrm{i}}} \mathbb{I},$$

$$\mathbf{s}^{\mathrm{i}}_{,\mathbf{B}} := \frac{q_{\mathrm{i}}}{m_{\mathrm{i}}} \mathbf{u}_{\mathrm{i}} \times \mathbb{I}.$$

(Similar relations hold for electrons.)

For the two-fluid dispersion relation, the uniform background state satisfies $\mathbf{u}_{\mathrm{i}} = \mathbf{u}_{\mathrm{e}} = \mathbf{u}_0$, and WLOG $\mathbf{u}_0 = 0$, so $\mathbf{s}^{\mathrm{i}}_{,\mathbf{B}} = 0 = \mathbf{s}^{\mathbf{E}}_{,\rho_{\mathrm{i}}}$. In this case, the velocity rescaling $\widetilde{\mathbf{u}}_\alpha := \mathbf{u}_\alpha / \mathbf{u}_{\alpha,0}$, where $\mathbf{u}_{\alpha,0} := \sqrt{\epsilon_0/\rho_\alpha}$, makes S_P antisymmetric, so all eigenvalues of S_P are imaginary. If $\mathbf{u}_{\mathrm{i}} \neq \mathbf{u}_{\mathrm{e}}$ then eigenvalues with positive real part give rise to a ***two-stream instability*** (See Nicholson §7.13).

Two-fluid waves

As in MHD, the waves in the two-fluid model are *determined by the choice of density, temperature, and magnetic field* of the background state and by the choice of wave number **k**.

But characterizing waves in the two-fluid model is much more complicated than for MHD. Most critically, wave speeds depend not only on the angle between **k** and the magnetic field, but also on the magnitude of **k**. A **dispersion diagram** (e.g. Figures 7.14 and 7.17 on page 159 of Nicholson) shows frequency as a function of wave number for some subset of the six wave speeds that arise in the two-fluid model.

Furthermore, unlike MHD, where the state of the plasma is characterized by one parameter (plasma β), in the two-fluid model the state of the plasma is characterized by three parameters:

1. *Plasma frequency* (or particle density),
2. *Gyrofrequency* (or magnetic field strength), and
3. *Plasma* β (or temperature).

In the *cold plasma approximation* ($\beta = 0$), the first two parameters are sufficient, and it is possible to indicate the topology of wave-normal surface on a two-dimensional diagram called a *Clemmow-Mullaly-Allis (CMA) diagram*.

1. Framework
   - Linearization
   - Waves and eigenstructure
2. MHD
   - waves
   - instabilities
3. Two-fluid
   - linearization
   - waves
   - instabilities

The two-fluid model simplifies the kinetic model only by assuming that the collision period is small and retains all other modeling parameters.

The two-fluid model gives a way to test when small scales that MHD neglects (e.g. plasma period, gyroperiod, or Debye length, but not mean free path) are needed.

In particular, the two-fluid model gives a way to test uniform background states that do not maximize entropy. Specifically, the **two-stream instability** arises due to relative drift between ions and electrons, as discussed in Nicholson Section 7.13. This instability has been proposed as a source of anomalously high resistivity in plasmas.

Reconnection

There are some important macroscale instabilities that cannot be properly resolved with MHD or a two-fluid model. One of the most important is *magnetic reconnection*. Fast magnetic reconnection triggers the most powerful and explosing space weather events in the solar system and is critical to the project of space weather forecasting. Linearized models have played an important role in understanding magnetic reconnection, but numerical simulation is proving necessary to get a coherent picture.

References

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