Fluid models of plasma

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Physical constants that define an ion-electron plasma:

1. e (charge of proton),
2. m_i, m_e (ion and electron mass),
3. c (speed of light),
4. ϵ_0 (vacuum permittivity).

Fundamental parameters that characterize the state of a plasma:

1. n_0 (typical particle density),
2. T_0 (typical temperature),
3. B_0 (typical magnetic field).

Derived quantities:

- $p_0 := n_0 T_0$ (thermal pressure)
- $p_B := \frac{B_0^2}{2\mu_0}$ (magnetic pressure)
- $\rho_s := n_0 m_s$ (typical density).

Collision periods:

- τ_{sp}: expected time for 90-degree deflection of species s via p.

Collisionless time, velocity, and space scale parameters:

plasma frequencies: $\omega_{p,s}^2 := \dfrac{n_0 e^2}{\epsilon_0 m_s}$,

gyrofrequencies: $\omega_{g,s} := \dfrac{eB_0}{m_s}$,

thermal velocities: $v_{t,s}^2 := \dfrac{2p_0}{\rho_s}$,

Alfvén speeds: $v_{A,s}^2 := \dfrac{2p_B}{\rho_s} = \dfrac{B_0^2}{\mu_0 m_s n_0}$,

Debye length: $\lambda_D := \dfrac{v_{t,s}}{\omega_{p,s}} = \sqrt{\dfrac{\epsilon_0 T_0}{n_0 e^2}}$,

gyroradii: $r_{g,s} := \dfrac{v_{t,s}}{\omega_{g,s}} = \dfrac{m_s v_{t,s}}{eB_0}$,

skin depths: $\delta_s := \dfrac{v_{A,s}}{\omega_{g,s}} = \dfrac{c}{\omega_{p,s}} = \sqrt{\dfrac{m_s}{\mu_0 n_s e^2}}$.

plasma $\beta := \dfrac{p_0}{p_B} = \left(\dfrac{v_{t,s}}{v_{A,s}}\right)^2 = \left(\dfrac{r_{g,s}}{\delta_s}\right)^2$.

non-MHD ratio: $\dfrac{c}{v_{A,s}} = \dfrac{r_{g,s}}{\lambda_D} = \dfrac{\omega_{p,s}}{\omega_{g,s}}$.

Plasma model hierarchy

1. **Particle Maxwell**: discrete particles: $(\mathbf{x}_p(t), \mathbf{v}_p(t))$

   $\downarrow$ large number of particles (per "mesh cell")

2. **Kinetic Maxwell**: particle density functions: $f_s(\mathbf{x}, \mathbf{v})$

   $\downarrow$ fast *collisions* $(\tau_{ss} \to 0)$.

3. **two-fluid Maxwell**: one gas for each species: $\rho_\mathrm{s}(\mathbf{x})$, $\mathbf{u}_\mathrm{s}(\mathbf{x})$, $e_\mathrm{s}(\mathbf{x})$

   $\downarrow$ fast light waves $(c \to \infty)$, charge *neutrality* $(\lambda_D \to 0)$.

4. **extended MHD**: gas that conducts electricity: $\rho(\mathbf{x})$, $\mathbf{u}(\mathbf{x})$, $e(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$;

   $$\mathbf{J} = \mu_0^{-1}\nabla \times \mathbf{B}, \quad \mathbf{E} = \mathbf{u} \times \mathbf{B} + \eta\mathbf{J} + \cdots.$$

   $\downarrow$ small gyroradius $(r_\mathrm{g} \to 0)$ and gyroperiod $(\omega_\mathrm{g} \to \infty)$.

5. **Ideal MHD**: a perfectly conducting gas: $\mathbf{E} = \mathbf{u} \times \mathbf{B}$.

Fundamental model: particle-Maxwell (relativistic)

Maxwell's equations:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0,$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\epsilon_0.$$

Charge moments:

$$\sigma := \sum_p S_p(\mathbf{x}_p) q_p,$$
$$\mathbf{J} := \sum_p S_p(\mathbf{x}_p) q_p v_p,$$

Particle equations:

$$d_t \mathbf{x}_p = \mathbf{v}_p,$$
$$d_t(\gamma_p \mathbf{v}_p) = \mathbf{a}_\mathrm{p}(\mathbf{x}_p, \mathbf{v}_p),$$
$$\gamma_p^{-2} := 1 - (\mathbf{v}_p/c)^2.$$

Lorentz acceleration:

$$\mathbf{a}_\mathrm{p}(\mathbf{x}, \mathbf{v}) = \frac{q_p}{m_p}(\mathbf{E}(\mathbf{x}) + \mathbf{v} \times \mathbf{B}(\mathbf{x}))$$

Changing SI to Gaussian units:

- replace **B** with $\mathbf{B}/c$.
- choose $\epsilon_0^{-1} = 4\pi$.

Problem: model based on particles is not a computationally accessible standard of truth for most applications.

Solution: replace particles with a particle density function $f_s(t, \mathbf{x}, \gamma\mathbf{v})$ for each species s.

2-species kinetic-Maxwell (relativistic)

Maxwell's equations:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0,$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\epsilon_0.$$

Charge moments:

$$\sigma := \sum_s \frac{q_s}{m_s} \int f_s \, d(\gamma \mathbf{v}),$$
$$\mathbf{J} := \sum_s \frac{q_s}{m_s} \int \mathbf{v} f_s \, d(\gamma \mathbf{v}).$$

Kinetic equations:

$$\boxed{\begin{aligned} \partial_t f_i + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + \mathbf{a}_i \cdot \nabla_{(\gamma \mathbf{v})} f_i &= \mathcal{C}_i \\ \partial_t f_e + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_e + \mathbf{a}_e \cdot \nabla_{(\gamma \mathbf{v})} f_e &= \mathcal{C}_e \end{aligned}}$$

Lorentz acceleration:

$$\mathbf{a}_i = \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$
$$\mathbf{a}_e = \frac{q_e}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

"Collision" operator

- includes all microscale effects
- conservation: $\int \mathbf{m}(\mathcal{C}_i + \mathcal{C}_e) \gamma^{-1} d(\gamma \mathbf{v}) = 0$, where $\mathbf{m} = (1, \gamma \mathbf{v}, \gamma)$.
- decomposed as:

  $$\mathcal{C}_i = \widetilde{\mathcal{C}}_{ii} + \overleftrightarrow{\mathcal{C}}_{ie},$$
  $$\mathcal{C}_e = \widetilde{\mathcal{C}}_{ee} + \overleftrightarrow{\mathcal{C}}_{ei},$$

  where $\int \mathbf{m} \widetilde{\mathcal{C}}_{ss} \gamma^{-1} d(\gamma \mathbf{v}) = 0$.
- "collisionless": $\overleftrightarrow{\mathcal{C}}_{sp} \approx 0$.

BGK collision operator

$$\widetilde{\mathcal{C}}_{ss} = \frac{\mathcal{M}_s - f_s}{\tau_{ss}},$$

where the entropy-maximizing distribution $\mathbf{M}$ shares physically conserved moments with f:

$$\mathcal{M} = \exp(\alpha \cdot \mathbf{m}),$$
$$\int \mathbf{m}(\mathcal{M} - f) d(\gamma \mathbf{v}) = 0.$$

Maxwell's equations:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0,$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\epsilon_0.$$

Charge moments:

$$\sigma := \sum_s \frac{q_s}{m_s} \int f_s \, d\mathbf{v},$$
$$\mathbf{J} := \sum_s \frac{q_s}{m_s} \int \mathbf{v} f_s \, d\mathbf{v}.$$

Kinetic equations:

$$\boxed{\begin{aligned} \partial_t f_i + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + \mathbf{a}_i \cdot \nabla_{\mathbf{v}} f_i &= \mathcal{C}_i \\ \partial_t f_e + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_e + \mathbf{a}_e \cdot \nabla_{\mathbf{v}} f_e &= \mathcal{C}_e \end{aligned}}$$

Lorentz acceleration:

$$\mathbf{a}_i = \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$
$$\mathbf{a}_e = \frac{q_e}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

"Collision" operator

- includes all microscale effects
- conservation: $\int_{\mathbf{v}} \mathbf{m}(\mathcal{C}_i + \mathcal{C}_e) = 0$, where $\mathbf{m} = (1, \mathbf{v}, \|\mathbf{v}\|^2)$.
- decomposed as:

$$\mathcal{C}_i = \widetilde{\mathcal{C}}_{ii} + \overleftrightarrow{\mathcal{C}}_{ie},$$
$$\mathcal{C}_e = \widetilde{\mathcal{C}}_{ee} + \overleftrightarrow{\mathcal{C}}_{ei},$$

  where $\int_{\mathbf{v}} \mathbf{m}\widetilde{\mathcal{C}}_{ii} = 0 = \int_{\mathbf{v}} \mathbf{m}\widetilde{\mathcal{C}}_{ee}$.

- "collisionless": $\overleftrightarrow{\mathcal{C}}_{sp} \approx 0$.

BGK collision operator

$$\widetilde{\mathcal{C}}_{ss} = \frac{\mathcal{M}_s - f_s}{\tau_{ss}},$$

where the Maxwellian distribution **M** shares physically conserved moments with f:

$$\mathcal{M} = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(\frac{-|\mathbf{c}|^2}{2\theta}\right),$$

$$\theta := \langle |\mathbf{c}|^2/2 \rangle.$$

Maxwell's equations:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0,$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\epsilon_0.$$

Charge moments:

$$\sigma := \sigma_\mathrm{i} + \sigma_\mathrm{e}, \quad \sigma_\mathrm{s} := \frac{q_\mathrm{s}}{m_\mathrm{s}} \rho_\mathrm{s}.$$
$$\mathbf{J} := \mathbf{J}_\mathrm{i} + \mathbf{J}_\mathrm{e}, \quad \mathbf{J}_\mathrm{s} := \sigma_\mathrm{s} \mathbf{u}_\mathrm{s}.$$

Evolved moments:

$$\begin{bmatrix} \rho_\mathrm{s} \\ \rho_\mathrm{s} \mathbf{u}_\mathrm{s} \\ \rho_\mathrm{s} e_\mathrm{s} \end{bmatrix} := \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}\|\mathbf{c}_\mathrm{s}\|^2 \end{bmatrix} f_\mathrm{s}\, d\mathbf{v}$$

Evolution equations:

$$\partial_t \rho_\mathrm{s} + \nabla \cdot (\mathbf{u}_\mathrm{s} \rho_\mathrm{s}) = 0,$$
$$\rho_\mathrm{s} d_t^\mathrm{s} \mathbf{u}_\mathrm{s} + \nabla p_\mathrm{s} + \nabla \cdot \mathbb{P}_\mathrm{s}^\circ = \sigma_\mathrm{s} \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \mathbf{R}_\mathrm{s}$$
$$\rho_\mathrm{s} d_t^\mathrm{s} e_\mathrm{s} + p_\mathrm{s} \nabla \cdot \mathbf{u}_\mathrm{s} + \mathbb{P}_\mathrm{s}^\circ : \nabla \mathbf{u}_\mathrm{s} + \nabla \cdot \mathbf{q}_\mathrm{s} = Q_\mathrm{s}$$

Closures (neglect):

$$\left[\begin{aligned}
\frac{\mathbf{R}_\mathrm{e}}{en} &\approx \boldsymbol{\eta} \cdot \mathbf{J} + \boldsymbol{\beta}_\mathrm{e} \cdot \mathbf{q}_\mathrm{e}, \\
\mathbf{R}_\mathrm{i} &= -\mathbf{R}_\mathrm{e}, \\
Q_\mathrm{s} &=: Q_\mathrm{s}^\mathrm{ex} + Q_\mathrm{s}^\mathrm{fr}, \\
Q_\mathrm{s}^\mathrm{ex} &\approx \tfrac{3}{2} K_\mathrm{s}\, n^2 (T_0 - T_s), \\
Q^\mathrm{fr} &:= Q_\mathrm{i}^\mathrm{fr} + Q_\mathrm{e}^\mathrm{fr} \\
&\approx \boldsymbol{\eta} : \mathbf{J}\mathbf{J} + \boldsymbol{\beta}_\mathrm{e} : \mathbf{q}_\mathrm{e} \mathbf{J}, \\
Q_\mathrm{i}^\mathrm{fr} &= Q_\mathrm{e}^\mathrm{fr} m_\mathrm{e}/m_\mathrm{i}, \\
\mathbb{P}_\mathrm{s}^\circ &\approx -2\boldsymbol{\mu}_\mathrm{s} : \nabla \mathbf{u}_\mathrm{s}^\circ, \\
\mathbf{q}_\mathrm{s} &\approx -\mathbf{k}_\mathrm{s} \cdot \nabla T_\mathrm{s}.
\end{aligned}\right]$$

Definitions:

$$d_t^\mathrm{s} := \partial_t + \mathbf{u}_s \cdot \nabla,$$
$$\mathbf{c}_\mathrm{s} := \mathbf{v} - \mathbf{u}_\mathrm{s},$$
$$n_\mathrm{s} := \rho_\mathrm{s}/m_\mathrm{s},$$
$$\mathbb{X}^\circ := \frac{\mathbb{X} + \mathbb{X}^\mathrm{T}}{2} - \frac{\mathbb{I}\,\mathrm{tr}\,\mathbb{X}}{3}.$$

Collisional sources:

$$\mathbf{R}_\mathrm{s} := \int \mathbf{v}\, \overleftrightarrow{\mathcal{C}_\mathrm{s}}\, d\mathbf{v},$$
$$Q_\mathrm{s} := \int \tfrac{1}{2}\|\mathbf{c}_\mathrm{s}\|^2\, \overleftrightarrow{\mathcal{C}_\mathrm{s}}\, d\mathbf{v}.$$

Closing moments (intraspecies):

$$\mathbb{P}_\mathrm{s} := \int \mathbf{c}_\mathrm{s} \mathbf{c}_\mathrm{s}\, f_\mathrm{s}\, d\mathbf{v},$$
$$p_\mathrm{s} := \tfrac{1}{3}\,\mathrm{tr}\,\mathbb{P}_\mathrm{s},$$
$$\mathbb{P}_\mathrm{s}^\circ := \mathbb{P}_\mathrm{s} - p_\mathrm{s}\mathbb{I},$$
$$\mathbf{q}_\mathrm{s} := \int \tfrac{1}{2}\mathbf{c}_\mathrm{s}\|\mathbf{c}_\mathrm{s}\|^2\, f_\mathrm{s}\, d\mathbf{v}.$$

2-fluid MHD (extended)

electromagnetism ($\partial_t \mathbf{E} \approx 0$)

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

Ohm's law (evolution of **J** solved for **E**)

$$\begin{aligned}\mathbf{E} &= \boldsymbol{\eta} \cdot \mathbf{J} + \mathbf{B} \times \mathbf{u} + \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B} \\ &+ \frac{1}{e\rho} \nabla \cdot \left(m_\mathrm{e}(p_\mathrm{i}\mathbb{I} + \mathbb{P}_\mathrm{i}^\circ) - m_\mathrm{i}(p_\mathrm{e}\mathbb{I} + \mathbb{P}_\mathrm{e}^\circ) \right) \\ &+ \frac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{u}\mathbf{J} + \mathbf{J}\mathbf{u} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J}\mathbf{J} \right) \right]\end{aligned}$$

mass and momentum (total):

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$
$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

energy evolution (per species):

$$\rho_\mathrm{i} d_t e_\mathrm{i} + p_\mathrm{i} \nabla \cdot \mathbf{u}_\mathrm{i} + \mathbb{P}_\mathrm{i}^\circ : \nabla \mathbf{u}_\mathrm{i} + \nabla \cdot \mathbf{q}_\mathrm{i} = Q_\mathrm{i},$$
$$\rho_\mathrm{e} d_t e_\mathrm{e} + p_\mathrm{e} \nabla \cdot \mathbf{u}_\mathrm{e} + \mathbb{P}_\mathrm{e}^\circ : \nabla \mathbf{u}_\mathrm{e} + \nabla \cdot \mathbf{q}_\mathrm{e} = Q_\mathrm{e};$$

Closures (simplified):

$$\begin{aligned} Q &:= Q_\mathrm{i} + Q_\mathrm{e} \\ &\approx \boldsymbol{\eta} : \mathbf{J}\mathbf{J} \\ Q_\mathrm{s} &= \frac{m_\mathrm{red}}{m_\mathrm{s}} Q, \\ \mathbb{P}_\mathrm{s}^\circ &\approx -2\boldsymbol{\mu}_\mathrm{s} : \nabla \mathbf{u}_\mathrm{s}^\circ, \\ \mathbf{q}_\mathrm{s} &\approx -\mathbf{k}_\mathrm{s} \cdot \nabla T_\mathrm{s}. \end{aligned}$$

Definitions:

$$\begin{aligned} d_t &:= \partial_t + \mathbf{u} \cdot \nabla, \\ \mathbf{w} &= \frac{\mathbf{J}}{en}, \\ \mathbf{w}_\mathrm{i} &= \frac{m_\mathrm{red}}{m_\mathrm{i}} \mathbf{w}, \\ \mathbf{w}_\mathrm{e} &= \frac{-m_\mathrm{red}}{m_\mathrm{e}} \mathbf{w}, \\ \mathbb{P}^\mathrm{d} &:= m_\mathrm{red} n \mathbf{w}\mathbf{w} \\ m_\mathrm{red}^{-1} &:= m_\mathrm{e}^{-1} + m_\mathrm{i}^{-1}. \end{aligned}$$

Resistive MHD

MHD system:

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0 \qquad \text{(mass continuity)},$$
$$\rho \mathrm{d}_t \mathbf{u} + \nabla p + \nabla\cdot\mathbb{P}^\circ = \mathbf{J}\times\mathbf{B} \qquad \text{(momentum balance)},$$
$$\partial_t \mathcal{E} + \nabla\cdot(\mathbf{u}(\mathcal{E}+p) + \mathbf{u}\cdot\mathbb{P}^\circ + \mathbf{q}) = \mathbf{J}\cdot\mathbf{E} \qquad \text{(energy balance)},$$
$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0 \qquad \text{(magnetic field evolution)}.$$

The divergence constraint $\nabla\cdot\mathbf{B} = 0$ is maintained by exact solutions and must be maintained in numerical solutions.

Electromagnetic closing relations:

$$\mathbf{J} := \mu_0^{-1}\nabla\times\mathbf{B} \qquad \text{(Ampere's law for current)},$$
$$\boxed{\mathbf{E} \approx \mathbf{B}\times\mathbf{u} + \eta\cdot\mathbf{J}} \qquad \text{(Ohm's law for electric field)}.$$

Fluid closure:

$$\mathbb{P}^\circ \approx -2\mu : \nabla\mathbf{u}^\circ,$$
$$\mathbf{q} \approx -\mathbf{k}\cdot\nabla T.$$

Descriptions:

- ρ = total mass density
- $\rho\mathbf{u}$ = total momentum density
- $\mathbf{u}$ = velocity of bulk fluid
- $\mathcal{E}$ = total gas-dynamic energy density
- p = total scalar pressure
- $\mathbb{P}^\circ$ = total deviatoric pressure
- $\nabla\mathbf{u}^\circ$ = deviatoric rate of "strain" (deformation)
- T = temperature
- $\mathbf{q}$ = total heat flux
- η = resistivity
- μ = viscosity
- $\mathbf{k}$ = heat conductivity

1. Presentation of plasma models

2. Derivation of plasma models
   - Kinetic
   - Two-fluid
   - MHD
   - Conclusion

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Conservation law framework

Definitions:

- t = **time**
- $\mathbf{X}$ = **position**
- $U(t, \mathbf{X})$ = **conserved quantity**
- $\mathbf{F}(t, \mathbf{X})$ = **flux function** (e.g. $\mathbf{F}(U)$).
- $S(t, \mathbf{X}) = 0$: production of U is *zero*.
- Ω = arbitrary region
- $\widehat{\mathbf{n}}$ = outward unit vector
- $\mathrm{d}\mathbf{A} = \widehat{\mathbf{n}}\mathrm{d}A$: surface element
- $\mathrm{d}\mathbf{A} \cdot \mathbf{F}(t, \mathbf{X})$ = flux rate of U out of surface element

Conservation law:

$$
\begin{aligned}
& (\forall\Omega) \quad d_t \int_\Omega U = -\oint_{\partial\Omega} \widehat{\mathbf{n}} \cdot \mathbf{F} \\
\iff & (\forall\Omega) \quad \int_\Omega (\partial_t U + \nabla \cdot \mathbf{F}) = 0 \\
\iff & \boxed{\partial_t U + \nabla \cdot \mathbf{F} = 0}.
\end{aligned}
$$

Balance law framework

Definitions:

- t = **time**
- $\mathbf{X}$ = **position**
- $U(t, \mathbf{X})$ = **conserved quantity**
- $\mathbf{F}(t, \mathbf{X})$ = **flux function** (e.g. $\mathbf{F}(U)$).
- $S(t, \mathbf{X})$ = **production** of U.
- Ω = arbitrary region
- $\widehat{\mathbf{n}}$ = outward unit vector
- $\mathrm{d}\mathbf{A} = \widehat{\mathbf{n}}\mathrm{d}A$: surface element
- $\mathrm{d}\mathbf{A} \cdot \mathbf{F}(t, \mathbf{X})$ = flux rate of U out of surface element

Balance law:

$$
\begin{aligned}
&(\forall\Omega) \quad d_t \int_\Omega U = -\oint_{\partial\Omega} \widehat{\mathbf{n}} \cdot \mathbf{F} + \int_\Omega S \\
\iff &(\forall\Omega) \quad \int_\Omega (\partial_t U + \nabla \cdot \mathbf{F} - S) = 0 \\
\iff &\boxed{\partial_t U + \nabla \cdot \mathbf{F} = S}.
\end{aligned}
$$

Transport Derivatives

Given:

- t = time
- $\mathbf{X}$ = position
- $\mathbf{V}(t, X)$ = velocity field
- $\alpha(t, \mathbf{x})$ = arbitrary function
- $\rho(t, \mathbf{x})$ = density convected by $\mathbf{V}$
- $\mathrm{d}_t := \frac{\mathrm{d}}{\mathrm{d}t}$
- $\boxed{\bar{\delta}_t\alpha := \partial_t\alpha + \nabla\cdot(\mathbf{V}\alpha)}$
  = **"transport derivative"** of α.
- $\boxed{\mathrm{d}_t\alpha := \partial_t\alpha + \mathbf{V}\cdot\nabla\alpha}$
  = **material derivative** of α.

Properties:

- $\boxed{\bar{\delta}_t\alpha = \mathrm{d}_t\alpha + \alpha\nabla\cdot\mathbf{V}}$.
- $\bar{\delta}_t(\alpha\beta) = \mathrm{d}_t(\alpha\beta) + (\nabla\cdot\mathbf{V})\alpha\beta$
  $= (\mathrm{d}_t\alpha)\beta + \alpha(\mathrm{d}_t\beta) + (\nabla\cdot\mathbf{V})\alpha\beta$
  $= (\bar{\delta}_t\alpha)\beta + \alpha(\mathrm{d}_t\beta)$.
- $\boxed{\bar{\delta}_t(\rho\beta) = \rho\mathrm{d}_t\beta}$.

Conservation of transported material:

$\rho(t, \mathbf{x})$ is transported by $\mathbf{V}$
$\iff$ $\mathbf{F} := \mathbf{V}\rho$ is a flux for ρ
$\iff$ $\boxed{\partial_t\rho + \nabla\cdot(\mathbf{V}\rho) = 0}$
$\iff$ $\bar{\delta}_t\rho = 0$
$\iff$ $d_t\rho + \rho\nabla\cdot\mathbf{V} = 0$
$\iff$ $d_t \ln\rho = -\nabla\cdot\mathbf{V}$.

Incompressible flow:

$\mathbf{V}$ is incompressible
$\iff$ $d_t\rho = 0$
$\iff$ $d_t \ln\rho = 0$
$\iff$ $\boxed{\nabla\cdot\mathbf{V} = 0}$
$\iff$ $d_t\alpha = \bar{\delta}_t\alpha \quad (\forall\alpha)$.

Given: $\Omega(t)$ = region convected by $\mathbf{V}$.

Recall: $\bar{\delta}_t\alpha := \partial_t\alpha + \nabla\cdot(\mathbf{V}\alpha)$.

Reynolds transport theorem:

$$\boxed{d_t \int_{\Omega(t)} \alpha = \int_{\Omega(t)} \bar{\delta}_t\alpha} \quad (\forall\alpha(t,\mathbf{x})).$$

Convective conservation law:

$$\begin{aligned} & d_t \int_{\Omega(t)} \rho = 0 \quad (\forall\Omega(t) \text{ convected by } \mathbf{V}) \\ \iff & \int_{\Omega(t)} \bar{\delta}_t\rho = 0 \quad (\forall\Omega(t) \text{ convected by } \mathbf{V}) \\ \iff & \bar{\delta}_t\rho = 0 \\ \iff & \boxed{\partial_t\rho + \nabla\cdot(\mathbf{V}\rho) = 0} \end{aligned}$$

Proof:

$$\begin{aligned} d_t \int_{\Omega(t)} \alpha &= \int_{\Omega(t)} \partial_t\alpha + \oint_{\partial\Omega} \widehat{\mathbf{n}}\cdot(\mathbf{V}\alpha) \\ &= \int_{\Omega(t)} (\partial_t\alpha + \nabla\cdot(\mathbf{V}\alpha)) = \int_{\Omega(t)} \bar{\delta}_t\alpha; \end{aligned}$$

the first equality can be justified by time splitting:

- $d_t \int_{\Omega(t)} \alpha$ = rate of change of amount of changing stuff in moving region $\Omega(t)$,
- $\int_\Omega \partial_t\alpha$ = rate of change of amount of changing stuff in frozen region Ω,
- $\oint_{\partial\Omega(t)} \widehat{\mathbf{n}}\cdot(\mathbf{V}\alpha)$ = rate of change of amount of frozen stuff in moving region $\Omega(t)$. □

Given:

- $\mathbf{x}$: position
- $\mathbf{v} = \dot{\mathbf{x}}$: velocity
- $\mathbf{a} = \dot{\mathbf{v}}$: acceleration
- $\tilde{f}_\mathrm{s}$: number distribution of species s.
- $\tilde{f}_\mathrm{s}(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$: number of particles of species s in a region of state space with volume $\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$.
- m_s: particle mass of species s
- q_s: particle charge of species s
- $f_\mathrm{s} = m_\mathrm{s}\tilde{f}_\mathrm{s}$: mass distribution of species s.
- $\mathbf{a}_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$: Lorentz acceleration.
- $\mathbf{X} := (\mathbf{x}, \mathbf{v})$: position in state space.
- $\mathbf{V} := \dot{\mathbf{X}} = (\mathbf{v}, \mathbf{a}_\mathrm{s})$: velocity in state space.

We suppress the species index s when focusing on one species.

Theorem: Lorentz acceleration implies incompressible flow in phase space.

- Incompressible means $\nabla_\mathbf{X} \cdot \mathbf{V} = 0$.
- $\nabla_\mathbf{X} \cdot \mathbf{V} = \nabla_\mathbf{x} \cdot \mathbf{v} + \nabla_\mathbf{v} \cdot \mathbf{a}$
- $\nabla_\mathbf{x} \cdot \mathbf{v} = 0$ because $\mathbf{x}$ and $\mathbf{v}$ are independent variables.
- $\nabla_\mathbf{v} \cdot \mathbf{E}(t, \mathbf{x}) = 0$ for same reason.
- So $\nabla_\mathbf{v} \cdot \mathbf{a} = \frac{q}{m}\frac{\partial}{\partial \mathbf{v}_i}\epsilon_{ijk}\mathbf{v}_j B_k(t, \mathbf{x}) = 0$.

Vlasov equation (conservation of particles):

$f(t, \mathbf{X})$ is transported by $\mathbf{V}$

$$\iff \partial_t f + \nabla_\mathbf{X} \cdot (\mathbf{V} f) = 0$$

$$\iff \boxed{\boxed{\partial_t f + \nabla_\mathbf{x} \cdot (\mathbf{v} f) + \nabla_\mathbf{v} \cdot (\mathbf{a} f) = 0}}$$

(conservation form)

$$\iff \boxed{\partial_t f + \mathbf{v} \cdot \nabla_\mathbf{x} f + \mathbf{a} \cdot \nabla_\mathbf{v} \cdot f = 0}$$

$$\iff \partial_t f + \mathbf{V} \cdot \nabla_\mathbf{X} f = 0$$

Remark: conservation form is preferred for taking fluid moments.

Collision operator

Kinetic equation = Vlasov with collisions:

$$\boxed{\partial_t f_s + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{a}_s f_s) = \mathcal{C}_s}$$

- The **collision operator** $\mathcal{C}_s(t, \mathbf{x}, \mathbf{v})$ represents evolution of f_s due to local collisions.
- $\mathcal{C}_i = \widetilde{\mathcal{C}}_{ii} + \overleftrightarrow{\mathcal{C}}_{ie}$, where the **intraspecies collision operator** $\widetilde{\mathcal{C}}_{ii}$ represents the effect of ion-ion collisions and the **interspecies collision operator** $\overleftrightarrow{\mathcal{C}}_{ie}$ represents the effect *on the ions* of ion-electron collisions.
- $\mathcal{C}_s$ is an *operator* which maps functions of velocity space, $f_i(\mathbf{v})$ and $f_e(\mathbf{v})$, to a function of velocity space, $\mathcal{C}_s(\mathbf{v})$.
- $\mathcal{C}_s$ is best understood in terms of *time splitting:* alternate between evolving the Vlasov equation and applying the collision operator at each point in space.

What constraints do collisions respect?

- Conservation of *mass*: $\int_{\mathbf{v}} \mathcal{C}_s = 0$.
- Conservation of *momentum*:
  - $\int_{\mathbf{v}} \mathbf{v} \widetilde{\mathcal{C}}_{ii} = 0$.
  - $\int_{\mathbf{v}} \mathbf{v}(\mathcal{C}_i + \mathcal{C}_e) = 0$,
- Conservation of *energy*:
  - $\int_{\mathbf{v}} \|\mathbf{v}\|^2 \widetilde{\mathcal{C}}_{ii} = 0$.
  - $\int_{\mathbf{v}} \|\mathbf{v}\|^2 (\mathcal{C}_i + \mathcal{C}_e) = 0$,
- Physical *entropy* is nondecreasing:
  - $-\int_{\mathbf{v}} \widetilde{\mathcal{C}}_{ii} \log \tilde{f}_i \geq 0$.
  - $-\int_{\mathbf{v}} m_i^{-1} \mathcal{C}_i \log \tilde{f}_i - \int_{\mathbf{v}} m_e^{-1} \mathcal{C}_e \log \tilde{f}_e \geq 0$.

Example: BGK collision operator:

$$\widetilde{\mathcal{C}}_{ss} = \frac{\mathcal{M}_s - f_s}{\tau_{ss}}.$$

- $\mathcal{M}_s$: *Maxwellian distribution*. Has the greatest possible physical entropy for a given mass, momentum, and energy density.
- τ_{ss} = **collision period**: time scale on which distribution relaxes to a Maxwellian distribution.

Collision operator: why needed

Why do we need a collision operator?

- To incorporate microscale effects, e.g.:
  - particle interactions mediated by a microscale electric field (known as *"Coulomb collisions"*) and
  - microscale wave-particle interactions.
- To justify fluid models:
  - Kinetic models agree with fluid models in a limit where the collision period approaches zero.

Why is a collision operator needed to incorporate microscale effects?

- The Vlasov equation agrees exactly with a particle model if f is understood to be a sum of Dirac delta functions (one for each particle). In this case, the Vlasov equation is instead referred to as the "**Klimontovich equation**".

  The fine-grain electromagnetic field detail needed in such a model usually makes it computationally intractible, which is the whole reason for introducing a kinetic model in the first place.

- Choose a *resolution scale* Δx large enough so that by averaging over this scale the distributions f and the electromagnetic field can be decomposed into a smoothly varying **macroscopic part** f^0 plus a **microscopic part** f^1. By definition, a kinetic equation evolves the macroscopic part.

- The *Vlasov equation* evolves the macroscopic part independently of the microscopic part.

- The *Kinetic equation* uses a **collision operator** to estimate the effect of the microscopic part on the evolution of the macroscopic part.

1. Presentation of plasma models

2. Derivation of plasma models
   - Kinetic
   - Two-fluid
   - MHD
   - Conclusion

Given definitions:

- $\chi(\mathbf{v}) = \begin{cases} 1 & \text{zeroth moment} \\ \mathbf{v} & \text{first moment} \\ v^2 & \text{second moment} \end{cases}$
- $\langle \chi \rangle_\mathrm{s} := \frac{\int_\mathbf{v} \chi f_\mathrm{s}}{\int_\mathbf{v} f_\mathrm{s}}$ is the statistical **mean** of χ for species s.
- $\rho_\mathrm{s} := \int_\mathbf{v} f_\mathrm{s}$ (mass density)
- $\boxed{\rho_\mathrm{s} \langle \chi \rangle_\mathrm{s} := \int_\mathbf{v} \chi f_\mathrm{s}}$. (generic moment)
- $\mathbf{u}_\mathrm{s} := \langle \mathbf{v} \rangle_\mathrm{s}$. (bulk velocity)
- $\mathbf{c}_\mathrm{s} := \mathbf{v} - \mathbf{u}_\mathrm{s}$. (thermal velocity)
- $n_\mathrm{s} = \frac{1}{m_\mathrm{s}} \rho_\mathrm{s}$ (number density)
- $\sigma_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}} \rho_\mathrm{s}$ (charge density)
- $\rho_\mathrm{s} \mathbf{u}_\mathrm{s}$ (momentum)
- *. . . dropping subscript* s*. . .*

Taking generic moment of the kinetic equation:

$$\int_\mathbf{v} \chi \Big(\partial_t f + \nabla_\mathbf{x} \cdot (\mathbf{v} f) + \nabla_\mathbf{v} \cdot (\mathbf{a} f) = \mathcal{C} \Big)$$

$$\iff \partial_t \int_\mathbf{v} \chi f + \int_\mathbf{v} \chi \nabla_\mathbf{x} \cdot (\mathbf{v} f) + \int_\mathbf{v} \chi \nabla_\mathbf{v} \cdot (\mathbf{a} f) = \int_\mathbf{v} \chi \mathcal{C}$$

$$\iff \partial_t \int_\mathbf{v} \chi f + \nabla_\mathbf{x} \cdot \left(\int_\mathbf{v} \mathbf{v} \chi f \right) = \int_\mathbf{v} \mathbf{a} \cdot (\nabla_\mathbf{v} \chi) f + \int_\mathbf{v} \chi \mathcal{C}$$

$$\iff \partial_t (\rho \langle \chi \rangle) + \nabla_\mathbf{x} \cdot (\rho \langle \mathbf{v} \chi \rangle) = \rho \langle \mathbf{a} \cdot \nabla_\mathbf{v} \chi \rangle + \int_\mathbf{v} \chi \mathcal{C}$$

$$\iff \boxed{\bar{\delta}_t (\rho \langle \chi \rangle) + \nabla_\mathbf{x} \cdot (\rho \langle \mathbf{c} \chi \rangle) = \rho \langle \mathbf{a} \cdot \nabla_\mathbf{v} \chi \rangle + \int_\mathbf{v} \chi \mathcal{C}}$$

Continuity equations:

- mass ($\chi = 1$):

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0}$$

- charge ($\chi = \frac{q}{m}$):

$$\partial_t \sigma + \nabla \cdot (\sigma \mathbf{u}) = 0$$

- number density ($\chi = \frac{1}{m}$):

$$\partial_t n + \nabla \cdot (n \mathbf{u}) = 0$$

Taking moments: momentum

Given definitions:

- *...dropping subscript* s...
- $\mathbf{u} := \langle \mathbf{v} \rangle$ (bulk velocity)
- $\mathbf{c} := \mathbf{v} - \mathbf{u}$ (thermal velocity)
- $\rho\mathbf{u}$ (momentum)
- $\mathbf{J} = \sigma\mathbf{u}$ (current)
- $\mathbf{R} := \int_{\mathbf{v}} \mathbf{c}\mathcal{C}$ (resistive drag)
- $\mathbb{P} := \rho\langle \mathbf{cc} \rangle$ (pressure tensor)

Relationships:

- $\mathbf{v} = \mathbf{u} + \mathbf{c}$, so
- $\langle \mathbf{c} \rangle = 0$
  (since $\langle \mathbf{c} \rangle = \langle \mathbf{v} \rangle - \mathbf{u} = 0$), so
- $\langle \mathbf{vc} \rangle = \langle \mathbf{cc} \rangle$
  (since $\langle \mathbf{uc} \rangle = \mathbf{u}\langle \mathbf{c} \rangle = 0$) and
- $\int_{\mathbf{v}} \mathbf{v}\mathcal{C} = \int_{\mathbf{v}} \mathbf{c}\mathcal{C}$
  (since $\int_{\mathbf{v}} \mathbf{u}\mathcal{C} = \mathbf{u}\int_{\mathbf{v}} \mathcal{C} = 0$).

Conservation of momentum ($\chi = \mathbf{v}$):

- Recall generic moment of the kinetic equation:

$$\bar{\delta}_t(\rho\langle \chi \rangle) + \nabla_{\mathbf{x}} \cdot (\rho\langle \mathbf{c}\chi \rangle) = \rho\langle \mathbf{a} \cdot \nabla_{\mathbf{v}}\chi \rangle + \int_{\mathbf{v}} \chi\mathcal{C}$$

- Using the relationships from the left column,

$$\bar{\delta}_t(\rho\mathbf{u}) + \nabla \cdot \mathbb{P} = \rho\langle \mathbf{a} \rangle + \mathbf{R}.$$

- But $\langle \mathbf{a} \rangle = \frac{q}{m}(\mathbf{E} + \mathbf{u} \times \mathbf{B})$. Thus:

$$\boxed{\bar{\delta}_t(\rho\mathbf{u}) + \nabla \cdot \mathbb{P} = \sigma\mathbf{E} + \mathbf{J} \times \mathbf{B} + \mathbf{R}}. \quad (1)$$

- **Kinetic energy balance** = momentum balance dot $\mathbf{u}$:

$$\rho \mathrm{d}_t(\tfrac{1}{2}\|\mathbf{u}\|^2) + \mathbf{u} \cdot (\nabla \cdot \mathbb{P}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{u} \cdot \mathbf{R}$$

- **Current balance** = momentum balance times $\frac{q}{m}$:

$$\frac{m}{q}\bar{\delta}_t\mathbf{J} + \nabla \cdot \mathbb{P} = \sigma\mathbf{E} + \mathbf{J} \times \mathbf{B} + \mathbf{R}.$$

Taking moments: energy

Given definitions:

- *... dropping subscript* s...
- $\mathcal{E} := \rho\langle \frac{1}{2} v^2 \rangle$ (energy density)
- $e := \langle \frac{1}{2} \|\mathbf{c}\|^2 \rangle$ (thermal energy per mass)
- $\mathbb{P} := \rho\langle \mathbf{cc} \rangle$ (pressure tensor)
- $\mathbf{q} := \rho\langle \frac{1}{2}\mathbf{c}\|\mathbf{c}\|^2 \rangle$ (heat flux)
- $Q := \int_{\mathbf{v}} \frac{1}{2}\|\mathbf{c}\|^2 \mathcal{C}$ (collisional heating)

Relationships:

- energy = kinetic plus thermal: $\langle \|\mathbf{v}\|^2 \rangle = \langle \|\mathbf{u}\|^2 \rangle + \langle \|\mathbf{c}\|^2 \rangle$, i.e., $\rho\langle \frac{1}{2}\|\mathbf{v}\|^2 \rangle = \rho\langle \frac{1}{2}\|\mathbf{u}\|^2 \rangle + \rho\langle \frac{1}{2}\|\mathbf{c}\|^2 \rangle$.

Energy balance:

- Recall generic moment evolution:

$$\bar{\delta}_t(\rho\langle\chi\rangle) + \nabla_{\mathbf{x}} \cdot (\rho\langle \mathbf{c}\chi \rangle) = \rho\langle \mathbf{a} \cdot \nabla_{\mathbf{v}}\chi \rangle + \int_{\mathbf{v}} \chi \mathcal{C}$$

- energy: $\chi = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$: using that:
  - $\rho\langle \frac{1}{2}\mathbf{cv}\cdot\mathbf{v} \rangle = \rho\langle \mathbf{cc} \rangle \cdot \mathbf{u} + \rho\langle \frac{1}{2}\mathbf{cc}\cdot\mathbf{c} \rangle = \mathbb{P}\cdot\mathbf{u} + \mathbf{q}$,
  - $\rho\langle \mathbf{a}\cdot\mathbf{v} \rangle = \rho\langle \frac{q}{m}\mathbf{E}\cdot\mathbf{v} \rangle = \mathbf{E}\cdot\frac{q}{m}\rho\mathbf{u} = \mathbf{E}\cdot\mathbf{J}$ (that is, $\langle \mathbf{a}\cdot\mathbf{v} \rangle = \langle \mathbf{a} \rangle \cdot \langle \mathbf{v} \rangle$),
  - $\int_{\mathbf{v}} \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\mathcal{C} = \int_{\mathbf{v}} \mathbf{u}\cdot\mathbf{v}\mathcal{C} + \int_{\mathbf{v}} \frac{1}{2}\mathbf{c}\cdot\mathbf{c}\mathcal{C} = \mathbf{R}\cdot\mathbf{u} + Q$

$$\boxed{\bar{\delta}_t\mathcal{E} + \nabla\cdot(\mathbb{P}\cdot\mathbf{u} + \mathbf{q}) = \mathbf{J}\cdot\mathbf{E} + \mathbf{R}\cdot\mathbf{u} + Q}$$

Thermal energy balance:

- Recall kinetic energy balance:

$$\bar{\delta}_t(\rho\tfrac{1}{2}\|\mathbf{u}\|^2) + \mathbf{u}\cdot(\nabla\cdot\mathbb{P}) = \mathbf{J}\cdot\mathbf{E} + \mathbf{R}\cdot\mathbf{u}$$

- Thermal energy balance equals energy balance minus kinetic energy balance:

$$\bar{\delta}_t(\rho\langle \tfrac{1}{2}\|\mathbf{c}\|^2 \rangle) + \mathbb{P} : \nabla\mathbf{u} + \nabla\cdot\mathbf{q} = Q$$

Full fluid equations (single fluid):

Restoring the species index s, we have a balance law for the mass(1) + momentum(3) + energy(1) = 5 conserved moments:

$$\boxed{\begin{aligned} &\bar{\delta}_t^{\mathrm{s}}\rho_{\mathrm{s}} && = 0 \\ &\bar{\delta}_t^{\mathrm{s}}(\rho_{\mathrm{s}}\mathbf{u}_{\mathrm{s}}) + \nabla\cdot\mathbb{P}_{\mathrm{s}} && = \sigma_{\mathrm{s}}\mathbf{E} + \mathbf{J}_{\mathrm{s}}\times\mathbf{B} + \mathbf{R}_{\mathrm{s}} \\ &\bar{\delta}_t^{\mathrm{s}}\mathcal{E}_{\mathrm{s}} \quad + \nabla\cdot(\mathbb{P}_{\mathrm{s}}\cdot\mathbf{u}_{\mathrm{s}} + \mathbf{q}_{\mathrm{s}}) && = \mathbf{J}_{\mathrm{s}}\cdot\mathbf{E} + \mathbf{R}_{\mathrm{s}}\cdot\mathbf{u}_{\mathrm{s}} + Q_{\mathrm{s}} \end{aligned}} \tag{2}$$

MHD fluid equations:

The bulk fluid quantities of MHD are defined by

$$\begin{aligned} \rho &:= \rho_{\mathrm{i}} \quad + \rho_{\mathrm{e}}, \\ \rho\mathbf{u} &:= \rho_{\mathrm{i}}\mathbf{u}_{\mathrm{i}} + \rho_{\mathrm{e}}\mathbf{u}_{\mathrm{e}}, \\ \mathcal{E} &:= \mathcal{E}_{\mathrm{i}} \quad + \mathcal{E}_{\mathrm{e}}. \end{aligned}$$

Summing each equation in System (2) over ions $(\mathrm{s} = \mathrm{i})$ and electrons $(\mathrm{s} = \mathrm{e})$ gives the corresponding MHD equation. So the MHD system is System (2) with the subscript s erased; in MHD, total charge σ is assumed to equal zero. The interspecies collision terms involving $\mathbf{R}_{\mathrm{s}}$ and Q_{s} cancel and disappear (why?).

Remarks

- System (2) is in the form

  $$\begin{aligned} &\bar{\delta}_t U + \nabla\cdot\widetilde{\mathbf{F}} = S, \quad \text{i.e.,} \\ &\partial_t U + \nabla\cdot(\mathbf{u}U + \widetilde{\mathbf{F}}) = S, \end{aligned}$$

  which is in the balance form

  $$\partial_t U + \nabla\cdot\mathbf{F} = S.$$

- In the MHD sum, we avoid introducing higher-order nonlinear terms by pretending that $\mathbf{u}_{\mathrm{s}} = \mathbf{u}$ and $\bar{\delta}_t^{\mathrm{s}} = \bar{\delta}_t$. In fact,

  $$\begin{aligned} \bar{\delta}_t^{\mathrm{i}} &:= \partial_t + \mathbf{u}_{\mathrm{i}}\cdot\nabla, \\ \bar{\delta}_t^{\mathrm{e}} &:= \partial_t + \mathbf{u}_{\mathrm{e}}\cdot\nabla, \quad \text{and} \\ \bar{\delta}_t &:= \partial_t + \mathbf{u}\cdot\nabla \end{aligned}$$

  are three (hopefully slightly) different things, because $\mathbf{u}_{\mathrm{s}} = \mathbf{u} + \mathbf{w}_{\mathrm{s}}$, where $\mathbf{w}_{\mathrm{s}}$ is the **drift velocity** of species s relative to the *bulk velocity* $\mathbf{u}$.

The pressure tensor is usually separated out into its scalar part $p_s = \frac{1}{3}\operatorname{tr}\mathbb{P}_s$ (which equals 2/3 the thermal energy) and its deviatoric (traceless) part $\mathbb{P}_s^\circ := \mathbb{P}_s - p_s\mathbb{I}$ (which cannot in general be inferred from the evolved moments). So more conventionally system (2) would be written:

$$\boxed{\begin{aligned}&\partial_t\rho_s + \nabla\cdot(\mathbf{u}_s\rho_s) = 0\\&\partial_t(\rho_s\mathbf{u}_s) + \nabla\cdot(\rho_s\mathbf{u}_s\mathbf{u}_s) + \nabla p_s + \nabla\cdot\mathbb{P}_s^\circ = \sigma_s\mathbf{E} + \mathbf{J}_s\times\mathbf{B} + \mathbf{R}_s\\&\partial_t\mathcal{E}_s + \nabla\cdot((\mathcal{E}_s + p_s)\mathbf{u}_s + \mathbb{P}_s^\circ\cdot\mathbf{u}_s + \mathbf{q}_s) = \mathbf{J}_s\cdot\mathbf{E} + \mathbf{R}_s\cdot\mathbf{u}_s + Q_s\end{aligned}} \tag{3}$$

The system (3) agrees exactly with the kinetic equation. The only problem is that it is not closed: the colored terms are unkown unless we make an assumption about the particle distribution. Fluid closures are derived by assuming that intraspecies collisions are fast enough to keep the distribution close to Maxwellian. If the distribution is Maxwellian then the red quantities, deviatoric pressure $\mathbb{P}_s^\circ$ and heat flux $\mathbf{q}_s$, will be zero. If there are no interspecies collisions between ions and electrons then the blue quantities, resistive drag $\mathbf{R}_s$ and collisional heating Q_s, will be zero.

1. Presentation of plasma models

2. Derivation of plasma models
   - Kinetic
   - Two-fluid
   - MHD
   - Conclusion

MHD models plasma as an *electrically conducting fluid*.

The (eXtended) MHD model simplifies the two-fluid model by making *two fundamental approximations*:

1. **Quasineutrality**: The net charge of both species is zero.
2. The displacement current $\partial_t \mathbf{E}$ is zero.

These approximations assume that the Debye length and plasma period are small (relative to the scales of interest). These are the smallest scales relevant in plasma modeling. MHD gives up on them.

Simplified versions of MHD result from additional approximations.

- **Two-fluid MHD** avoids additional assumptions.
- **One-fluid MHD** assumes that the drift velocity of electrons relative to ions is small.
- **Hall MHD** assumes that the electron gyroradius and gyroperiod are small.
- **Ideal MHD** assumes that all plasma modeling parameters are small, including ion gyroradius and gyroperiod and ion collision period.

MHD: bulk fluid quantities

Bulk versus two-fluid quantities:

- MHD evolves **bulk quantities**
  - $\rho := \rho_\mathrm{i} + \rho_\mathrm{e}$.
  - $\rho\mathbf{u} := \rho_\mathrm{i}\mathbf{u}_\mathrm{i} + \rho_\mathrm{e}\mathbf{u}_\mathrm{e}$.
  - $\mathcal{E} := \mathcal{E}_\mathrm{i} + \mathcal{E}_\mathrm{e}$.
- Heat flux:
  - $\mathbf{q}^\mathrm{g} := \mathbf{q}_\mathrm{i} + \mathbf{q}_\mathrm{e}$.
- *Quasineutrality* allows drift velocity to be inferred from current (because quasineutrality implies that current is independent of reference frame):
  - $0 \approx \sigma := \sigma_\mathrm{i} + \sigma_\mathrm{e}$.
  - $\mathbf{w}_\mathrm{s} := \mathbf{u}_\mathrm{s} - \mathbf{u}$.
  - $\mathbf{w} := \mathbf{w}_\mathrm{i} - \mathbf{w}_\mathrm{e}$.
  - $\mathbf{J} = \sigma_\mathrm{i}\mathbf{w} = \sigma_\mathrm{e}\mathbf{w}$ (because current is independent of reference frame)
  - $0 = m_\mathrm{e}\mathbf{w}_\mathrm{e} + m_\mathrm{i}\mathbf{w}_\mathrm{i}$. (by conservation of mass and charge neutrality)

MHD pressure. MHD has three kinds of pressure, due to gas pressure, interspecies drift, and magnetic field:

1. Gas pressure:
   - $p^\mathrm{g} := p_\mathrm{i} + p_\mathrm{e}$ is the *summed gas-dynamic pressure*.
   - $\mathbb{P}^\mathrm{g} := \mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e}$ is the *summed gas-dynamic pressure tensor*.
2. Interspecies drift:
   - $m_\mathrm{red}^{-1} := m_\mathrm{i}^{-1} + m_\mathrm{e}^{-1}$ (**reduced mass**),
   - $\mathbb{P}^\mathrm{d} := m_\mathrm{red} n\mathbf{w}\mathbf{w}$ (*"drift pressure tensor"*),
   - $p^\mathrm{d} := \frac{1}{3} m_\mathrm{red} n|\mathbf{w}|^2$ (*"drift pressure"*),
   - $p := p^\mathrm{g} + p^\mathrm{d}$ (*MHD gas-dynamic pressure*), and
   - $\mathbb{P} := \mathbb{P}^\mathrm{g} + \mathbb{P}^\mathrm{d}$ (*MHD gas-dynamic pressure tensor*)

   are defined so that gas-dynamic energy satisfies

   $$\mathcal{E} = \frac{3}{2}p + \frac{1}{2}\rho|\mathbf{u}|^2.$$

   (The drift pressure can be reliably neglected.)
3. $p^\mathrm{MHD} = p + p^\mathrm{B}$ (*total MHD pressure*) includes the **magnetic pressure** $p^\mathrm{B} := \frac{|\mathbf{B}|^2}{2\mu_0}$, defined by its ability to balance gas-dynamic pressure in steady-state solutions.

MHD gas dynamics (bulk fluid evolution)

Full fluid equations (one species):

Summing the equations in System (2) (see page 24) over both species gives conservation laws for density of total mass, momentum, and energy:

$$\boxed{\begin{aligned}\bar{\delta}_t\rho &= 0,\\ \bar{\delta}_t(\rho\mathbf{u}) + \nabla\cdot(\mathbb{P}^{\mathrm{g}} + \mathbb{P}^{\mathrm{d}}) &= \mathbf{J}\times\mathbf{B},\\ \bar{\delta}_t\mathcal{E} + \nabla\cdot(\mathbb{P}^{\mathrm{g}}\cdot\mathbf{u} + \mathbf{q}^{\mathrm{g}} + \mathbf{q}^{\mathrm{d}}) &= \mathbf{J}\cdot\mathbf{E}.\end{aligned}} \tag{4}$$

- $\mathbb{P}^{\mathrm{d}}$ and $\mathbf{q}^{\mathrm{d}}$ are trash bins for "bad" (nonlinear) terms (see right column). MHD throws them away.
- If the trash is retained, this simplified system agrees exactly with the two-fluid equations.
- *Problem:* even taking out the trash, this system is not closed:
  - What is $\mathbf{J}$?
  - What is σ?
- *Solution:* modify Maxwell: ($\cancel{\partial_t\mathbf{E}}$, $\sigma = 0$)...

Drift:

- Define $\mathbf{w}_{\mathrm{s}} := \mathbf{u}_{\mathrm{s}} - \mathbf{u}$ to be the **drift velocity** of species s.
- $\mathbb{P}^{\mathrm{d}} := \sum_{\mathrm{s}} \rho_{\mathrm{s}}\mathbf{w}_{\mathrm{s}}\mathbf{w}_{\mathrm{s}}$ is the **"drift pressure"**.
- $\mathbf{q}^{\mathrm{d}} := \sum_{\mathrm{s}} (\mathbf{w}_{\mathrm{s}}\mathcal{E}_{\mathrm{s}} + \mathbf{w}_{\mathrm{s}}\cdot\mathbb{P}_{\mathrm{s}})$ is the **"drift heat flux"**.
- Throwing away $\mathbb{P}^{\mathrm{d}}$ is safe[1], because $\mathbf{w}_{\mathrm{i}}$ and ρ_{e} are both relatively small.
- Throwing away $\mathbf{q}^{\mathrm{d}}$ is dangerous because $\mathbf{w}_{\mathrm{e}}$ could be large and because electron and ion pressure (or temperature or thermal energy) are comparable.
- When $\mathbf{q}^{\mathrm{d}}$ can be relatively large, retain separate energy equations (e.g., use *"two-fluid"* (two-temperature) *MHD* instead).

[1] except for pair plasma

MHD assumes that the light speed is infinite. This implies **quasineutrality:** that the net charge density is zero. Indeed, Maxwell's equations simplify to:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \cancel{c^{-2}\partial_t \mathbf{E}}, \qquad \mu_0 \sigma = 0 + \cancel{c^{-2}\nabla \cdot \mathbf{E}}.$$

This system is *Galilean-invariant*, and its relationship to gas-dynamics is fundamentally different:

variable	**MHD**	**2-fluid-Maxwell**
$\mathbf{J}$	$\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ (comes from **B**)	$\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e)$ (from gas dynamics)
σ	$\sigma = 0$ (quasineutrality) (gas-dynamic constraint)	$\sigma = e(n_i - n_e)$ (electric field constraint)
$\mathbf{E}$	supplied by *Ohm's law* (from gas dynamics)	evolved (from **B** and **J**)

MHD assumes **charge neutrality** ("**quasineutrality**"[a]): $0 = \sigma_{\mathrm{i}} + \sigma_{\mathrm{e}}$.

- Quasineutrality is valid on time scales greater than the electron plasma period and on spatial scales greater than a Debye length.
- Quasineutrality allows to infer the drift velocities $\mathbf{w}_{\mathrm{s}} := \mathbf{u}_{\mathrm{s}} - \mathbf{u}$ of two species from their net mass, momentum, and current densities.
    - Assume $q_{\mathrm{i}} = e$, $q_{\mathrm{e}} = -e$.
    - Then $n_e = n_i := n_0$.
    - Formulas for drift velocities:[b]

$$
\boxed{
\begin{aligned}
\mathbf{w}_{\mathrm{i}} &= \frac{m_{\mathrm{e}}}{m_{\mathrm{i}} + m_{\mathrm{e}}}\mathbf{w} \approx 0, && (5)\\
\mathbf{w}_{\mathrm{e}} &= \frac{-m_{\mathrm{i}}}{m_{\mathrm{i}} + m_{\mathrm{e}}}\mathbf{w} \approx -\mathbf{w}, && (6)\\
\mathbf{w} &= \frac{\mathbf{J}}{en_0}. && (7)
\end{aligned}
}
$$

Justification:

Formulas (5)–(6) are the solution to the linear system

$$
\begin{aligned}
0 &= m_{\mathrm{i}}\mathbf{w}_{\mathrm{i}} + m_{\mathrm{e}}\mathbf{w}_{\mathrm{e}} && \text{(momentum)},\\
\mathbf{w} &:= \mathbf{w}_{\mathrm{i}} - \mathbf{w}_{\mathrm{e}} && \text{(relative drift def.)},
\end{aligned}
$$

where the momentum relation holds because total momentum is zero in the reference frame of the fluid: $0 = m_{\mathrm{i}} n_0 \mathbf{w}_{\mathrm{i}} + m_{\mathrm{e}} n_0 \mathbf{w}_{\mathrm{e}}$. Formula (7) holds by choosing to measure the current alternately in the reference frame of the ions or electrons, because for a charge-neutral plasma current is the same in any two reference frames: $\mathbf{J} := \sum_{\mathrm{s}} (\mathbf{u} + \mathbf{w}_{\mathrm{s}})\,\sigma_{\mathrm{s}} = \mathbf{u}\left(\sum_{\mathrm{s}} \sigma_{\mathrm{s}}\right) + \sum_{\mathrm{s}} \mathbf{w}_{\mathrm{s}}\sigma_{\mathrm{s}} = \sum_{\mathrm{s}} \mathbf{w}_{\mathrm{s}}\sigma_{\mathrm{s}}$.

---

[a] Classical MHD assumes *exact* charge neutrality. The word quasineutrality is preferred by physicists who can't quite bring themselves to pretend that the speed of light is infinite.

[b] If $|q_{\mathrm{i}}| \neq |q_{\mathrm{e}}|$ then make the replacements $m_{\mathrm{s}} \to |\widetilde{m}_{\mathrm{s}}| := \frac{m_{\mathrm{s}}}{|q_{\mathrm{s}}|}$ and $en_0 \to \sigma_{\mathrm{e}}$

MHD: Ohm's law

Ohm's law provides a closure for **E** by solving electron momentum evolution for the electric field.

Recall from page 22 the momentum evolution equation (1). For electrons it says:

$$\bar{\delta}_t(\rho_e \mathbf{u}_e) + \nabla \cdot \mathbb{P}_e = \sigma_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \mathbf{R}_e.$$

The **resistive Ohm's law** assumes equilibrium and therefore discards all the differentiated quantities on the left hand side and solves for **E**:

$$\mathbf{E} = \mathbf{B} \times \mathbf{u}_e + \frac{\mathbf{R}_e}{\sigma_e}.$$

We assume that resistive drag is proportional to current: $\frac{\mathbf{R}_e}{\sigma_e} = -\boldsymbol{\eta} \cdot \mathbf{J}$. Resistive MHD assumes that drift velocity is small: $\mathbf{u}_e \approx \mathbf{u}$. More generally, from the previous slide we have that $\mathbf{u}_e = \mathbf{u} + \mathbf{w}_e \approx \mathbf{u} - \frac{\mathbf{J}}{en}$, so we get:

$$\boxed{\begin{aligned} \mathbf{E} =& \mathbf{B} \times \mathbf{u} && \text{(ideal term)} \\ &+ \frac{1}{en}\mathbf{J} \times \mathbf{B} && \text{(Hall term)} \\ &+ \boldsymbol{\eta} \cdot \mathbf{J} && \text{(resistive term)}, \end{aligned}}$$

where the Hall term comes from electron drift velocity and is inferred using the quasineutrality relations (6)–(7) on the previous slide: $\mathbf{u}_e = \mathbf{u} + \mathbf{w}_e \approx \mathbf{u} + \mathbf{w} = \mathbf{u} - \frac{\mathbf{J}}{en_0}$.

Ideal MHD keeps only the ideal term.

Putting it all together, we have. . .

MHD system:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{(mass continuity)},$$
$$\rho \mathrm{d}_t \mathbf{u} + \nabla p + \nabla \cdot \mathbb{P}^\circ = \mathbf{J} \times \mathbf{B} \qquad \text{(momentum balance)},$$
$$\bar{\delta}_t \mathcal{E} + \nabla \cdot (\mathbf{u} p + \mathbf{u} \cdot \mathbb{P}^\circ + \mathbf{q}) = \mathbf{J} \cdot \mathbf{E} \qquad \text{(energy balance)},$$
$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \qquad \text{(magnetic field evolution)}.$$

The divergence constraint $\nabla \cdot \mathbf{B} = 0$ is maintained by exact solutions and must be maintained in numerical solutions.

Electromagnetic closing relations:

$$\mathbf{J} := \mu_0^{-1} \nabla \times \mathbf{B} \qquad \text{(Ampere's law for current)}$$
$$\boxed{\mathbf{E} \approx \mathbf{B} \times \mathbf{u} + \boldsymbol{\eta} \cdot \mathbf{J}} \qquad \text{(Ohm's law for electric field)}$$

In a reference frame moving with the fluid, $\mathbf{B}$ remains unchanged but the electric field becomes $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} = \boldsymbol{\eta} \cdot \mathbf{J}$. So Ohm's law says that in the reference frame of the fluid, the electric field is proportional to current (i.e. to the drift velocity of the electrons). In other words, the electric field balances the resistive drag force.

Fluid closure:

$$\mathbb{P}^\circ \approx -2\boldsymbol{\mu} : \nabla \mathbf{u}^\circ,$$
$$\mathbf{q} \approx -\mathbf{k} \cdot \nabla T.$$

Remarks:

We will neglect the viscosity $\boldsymbol{\mu}$ and heat conductivity $\mathbf{k}$. In the presence of a strong magnetic field, $\boldsymbol{\mu}$ and $\mathbf{k}$ are tensors, not scalars! In a tokamak ("fusion doughnut"), heat conductivity perpendicular to the magnetic field can be a million times weaker than parallel to the magnetic field! (That's a good thing, since the whole point of a tokamak is to confine heat.) The reason is that particles spiral tightly around magnetic field lines; viewed on a large scale, they naturally drift along field lines, but they can be induced to move across field lines only with great difficulty.

On the other hand, even when the magnetic field is strong, it is safe to assume that the resistivity $\boldsymbol{\eta}$ is a scalar (i.e., $\boldsymbol{\eta} = \eta \mathbb{I}$) and we will make this simplification.

Conservation form of MHD

A fundamental principle of physics is that total momentum and energy are conserved. This means that we should be able to put e.g. the momentum evolution equation in conservation form $\partial_t Q + \nabla\cdot\mathbf{F} = 0$.

To put **momentum evolution** in conservation form, we write the source term as a divergence using Ampere's law, vector calculus, and $\nabla\cdot\mathbf{B} = 0$:

$$\begin{aligned}
&-\mu_0\mathbf{J}\times\mathbf{B} = \mu_0\mathbf{B}\times\mathbf{J}\\
&= \mathbf{B}\times\nabla\times\mathbf{B}\\
&= (\nabla\mathbf{B})\cdot\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{B}\\
&= \nabla(\mathbf{B}^2/2) - \nabla\cdot(\mathbf{B}\mathbf{B})\\
&= \nabla\cdot(\mathbb{I}\mathbf{B}^2/2 - \mathbf{B}\mathbf{B}).
\end{aligned}$$

To put **energy evolution** in conservation form, we write the source term as a time-derivative plus a divergence, using Ampere's law, the identity $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{B}$, and Faraday's law:

$$\begin{aligned}
&-\mu_0\mathbf{E}\cdot\mathbf{J}\\
&= -\mathbf{E}\cdot\nabla\times\mathbf{B}\\
&= \nabla\cdot(\mathbf{E}\times\mathbf{B}) - \mathbf{B}\cdot\nabla\times\mathbf{E}\\
&= \nabla\cdot(\mathbf{E}\times\mathbf{B}) + \mathbf{B}\cdot\partial_t\mathbf{B}\\
&= \nabla\cdot(\mathbf{E}\times\mathbf{B}) + \partial_t(\mathbf{B}^2/2).
\end{aligned}$$

So **MHD in conservation form** reads

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0 && \text{(mass continuity)},\\
&\rho\mathrm{d}_t\mathbf{u} + \nabla\cdot\left(\mathbb{I}\left(p + \tfrac{B^2}{2\mu_0}\right) + \mu_0^{-1}\mathbf{B}\mathbf{B} + \mathbb{P}^\circ\right) = 0, && \text{(momentum conservation)},\\
&\partial_t\left(\mathcal{E} + \tfrac{\mathbf{B}^2}{2\mu_0}\right) + \nabla\cdot\left(\mathbf{u}(\mathcal{E}+p) + \mathbf{u}\cdot\mathbb{P}^\circ + \mathbf{q} + \mu_0^{-1}\mathbf{E}\times\mathbf{B}\right) = 0, && \text{(energy conservation)},\\
&\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0 && \text{(magnetic field evolution)},
\end{aligned}$$

where we now recognize $p_B := \frac{\mathbf{B}^2}{2\mu_0}$ as both the pressure and the energy of the magnetic field.

Thermal energy evolution in MHD

To obtain a thermal energy evolution equation for MHD, we imitate the procedure for gas dynamics by subtracting kinetic energy evolution from total gas dynamic energy evolution.

Recall momentum balance:

$$\rho d_t \mathbf{u} + \nabla p + \nabla \cdot \mathbb{P}^\circ = \mathbf{J} \times \mathbf{B}.$$

Kinetic energy balance is **u** dot momentum balance:

$$\frac{1}{2}\rho d_t |\mathbf{u}|^2 + \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot (\nabla \cdot \mathbb{P}^\circ) = \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}).$$

Recall total gas-dyanamic energy balance:

$$\bar{\delta}_t \mathcal{E} + \nabla \cdot (\mathbf{u}p + \mathbf{u} \cdot \mathbb{P}^\circ + \mathbf{q}) = \mathbf{J} \cdot \mathbf{E}.$$

Subtracting kinetic energy balance from this yields thermal energy balance:

$$\boxed{\frac{3}{2}\bar{\delta}_t p + p\nabla \cdot \mathbf{u} + \mathbb{P}^\circ : \nabla \mathbf{u} + \nabla \cdot \mathbf{q} = \mathbf{J} \cdot \mathbf{E}'},$$

where $\mathbf{E}' := \mathbf{E} + \mathbf{u} \times \mathbf{B}$ is the electric field in the reference frame of the fluid. Here we have used the ideal gas law

$$\mathcal{E} = \frac{3}{2}p + \frac{1}{2}\rho|\mathbf{u}|^2,$$

which is presumed to hold for MHD.

For ideal MHD, $0 = \mathbf{E}' = \mathbb{P}^\circ = \mathbf{q}$, and it is common to write

$$\boxed{d_t p + \gamma p \nabla \cdot \mathbf{u} = 0}$$

where

$$\gamma := \frac{5}{3}$$

is the adiabatic index.

Net current evolution is a weighted sum of the momentum equations for electrons *and* for ions.

- For each species, multiplying momentum evolution by the charge to mass ratio yields **current evolution**:

$$\partial_t \mathbf{J}_\mathrm{s} + \nabla\cdot(\mathbf{u}_s \mathbf{J}_\mathrm{s}) + \nabla\cdot\left(\frac{q_\mathrm{s}}{m_\mathrm{s}}\mathbb{P}_\mathrm{s}\right) = \frac{q_\mathrm{s}}{m_\mathrm{s}}\left(\sigma_\mathrm{s}\mathbf{E} + \mathbf{J}_\mathrm{s}\times\mathbf{B}\right) + \frac{q_\mathrm{s}}{m_\mathrm{s}}\mathbf{R}_\mathrm{s}.$$

- Summing over both species and using charge neutrality gives **net current evolution**:

$$\boxed{\partial_t \mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho}\mathbf{JJ}\right) + e\nabla\cdot\left(\frac{\mathbb{P}_\mathrm{i}}{\mathbf{m}_\mathrm{i}} - \frac{\mathbb{P}_\mathrm{e}}{\mathbf{m}_\mathrm{e}}\right) = \frac{e^2\rho}{m_\mathrm{i} m_\mathrm{e}}\left(\mathbf{E} + \left(\mathbf{u} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho}\mathbf{J}\right)\times\mathbf{B} - \frac{\mathbf{R}_\mathrm{e}}{en}\right)},$$

  where we have assumed the quasineutrality relations $\sigma_\mathrm{i} = en$ and $\sigma_\mathrm{e} = -en$ and $\rho = n(m_\mathrm{i} + m_\mathrm{e})$, and where we have used that $\mathbf{R}_\mathrm{i} = -\mathbf{R}_\mathrm{e}$. For the inertial term we have used $\mathbf{J}_\mathrm{s} = \mathbf{u}_\mathrm{s}\sigma_\mathrm{s}$ and $\sum_\mathrm{s}\mathbf{u}_\mathrm{s}\mathbf{J}_\mathrm{s} = \sum_\mathrm{s}\mathbf{u}_\mathrm{s}\mathbf{u}_\mathrm{s}\sigma_\mathrm{s} = \mathbf{Ju} + \mathbf{uJ} + \sum_\mathrm{s}\mathbf{w}_\mathrm{s}\mathbf{w}_\mathrm{s}\sigma_\mathrm{s}$, where $\sum_\mathrm{s}\mathbf{w}_\mathrm{s}\mathbf{w}_\mathrm{s}\sigma_\mathrm{s} = \frac{m_\mathrm{e} - m_\mathrm{i}}{m_\mathrm{e} + m_\mathrm{i}}\frac{\mathbf{JJ}}{en}$ follows from the quasineutral drift velocity relations $\mathbf{w} = \frac{\mathbf{J}}{ne}$, $\mathbf{w}_\mathrm{i} = \frac{m_\mathrm{e}}{m_\mathrm{e} + m_\mathrm{i}}\mathbf{w}$, and $\mathbf{w}_\mathrm{e} = \frac{-m_\mathrm{i}}{m_\mathrm{e} + m_\mathrm{i}}\mathbf{w}$.

- Solving for electric field yields **Ohm's law**. . .

Full Ohm's law

From the previous slide, net current evolution is

$$\partial_t \mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho}\mathbf{JJ}\right) + e\nabla\cdot\left(\frac{\mathbb{P}_\mathrm{i}}{\mathbf{m}_\mathrm{i}} - \frac{\mathbb{P}_\mathrm{e}}{\mathbf{m}_\mathrm{e}}\right) = \frac{e^2\rho}{m_\mathrm{i} m_\mathrm{e}}\left(\mathbf{E} + \left(\mathbf{u} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho}\mathbf{J}\right)\times\mathbf{B} - \frac{\mathbf{R}_\mathrm{e}}{en}\right)$$

A closure for the collisional term is $\frac{\mathbf{R}_\mathrm{e}}{en} = \boldsymbol{\eta}\cdot\mathbf{J} + \boldsymbol{\beta}_\mathrm{e}\cdot\mathbf{q}_\mathrm{e}$.

Ohm's law is current evolution solved for the electric field:

$$\boxed{\begin{aligned}
\mathbf{E} =& \mathbf{B}\times\mathbf{u} && \text{(ideal term)}\\
&+ \tfrac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{J}\times\mathbf{B} && \text{(Hall term)}\\
&+ \boldsymbol{\eta}\cdot\mathbf{J} && \text{(resistive term)}\\
&+ \boldsymbol{\beta}_\mathrm{e}\cdot\mathbf{q}_\mathrm{e} && \text{(thermoelectric term)}\\
&+ \tfrac{1}{e\rho}\nabla\cdot\left(m_\mathrm{e}\mathbb{P}_\mathrm{i} - m_\mathrm{i}\mathbb{P}_\mathrm{e}\right) && \text{(pressure term)}\\
&+ \tfrac{m_\mathrm{i} m_\mathrm{e}}{e^2\rho}\left[\partial_t\mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \tfrac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{JJ}\right)\right] && \text{(inertial term).}
\end{aligned}}$$

Ohm's law gives an implicit closure to the induction equation, $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$ (so retaining the inertial term entails an implicit numerical method).

Model justification

Why are fluid models good?

- The mass, momentum, and energy moments are physically conserved.
- Maxwell's equations are defined in terms of fluid moments.
- Therefore, if we can accurately evolve moments, we don't need the detail of the kinetic distribution.

When are simplified models good?

- Kinetic models are good when the space-time box defined by the smallest scale of interest contains enough particles.
- Fluid models are good when the space-time box defined by the smallest scale of interest is big enough that the particle distribution is close to Maxwellian.
- The MHD assumption of quasineutrality is good on scales larger than a Debye length (so for any scale where a fluid model is relevant).
- Ideal MHD is good if all plasma modeling scales (see slide 3, "Modeling parameters") are smaller than the smallest scale of interest.

References

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