A case for 13-moment two-fluid MHD

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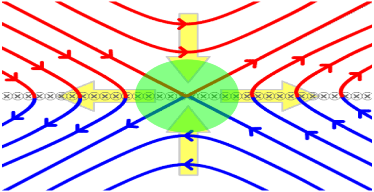
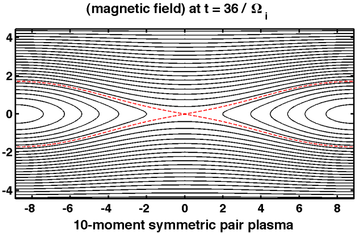




Outline

The purpose of this talk is to build a case that a 13-moment 2-fluid model is the simplest fluid model of plasma that can resolve steady fast magnetic reconnection and avoid anomalous cross-field transport in a highly magnetized plasma.

The argument:

1. A study of the terms of the XMHD Ohm's law and entropy evolution at the X-point of steady 2D reconnection invariant under 180-degree rotation reveals that nonzero heat flux and viscosity are model requirements.
2. In a highly magnitized plasma, anomalous cross-field transport is difficult to avoid unless spatially higher-order-accurate methods are used. Higher-order-accurate positivity-preserving methods are available for hyperbolic models but not yet for diffusive models. Relaxation closures are non-diffusive and trivial for the 13-moment model and are independent of the magnetic field. In contrast, relaxation closures for quadratic-moment models are diffusive, become complicated in the presence of a magnetic field, and become ill-conditioned when the magnetic field becomes strong. As a consequence, implicit methods are necessary and it is difficult to design appropriate preconditioners, particularly for positivity-preserving methods.

Model hierarchy

$$
\begin{tikzcd}
\text{(Two-species kinetic-Maxwell)} \arrow[d] & \\
\text{(13-moment two-fluid Maxwell)} \arrow[r] \arrow[d] & \text{(13-moment two-fluid MHD)} \arrow[d] \\
\text{(10-moment two-fluid Maxwell)} \arrow[r] \arrow[d] & \text{(10-moment two-fluid MHD)} \arrow[d] \\
\text{(5-moment two-fluid Maxwell)} \arrow[r] & \text{(5-moment two-fluid MHD)} \arrow[d] \\
 & \text{(5-moment MHD)}
\end{tikzcd}
$$

2-species kinetic-Maxwell with Gaussian-BGK collision operator

- **Kinetic equations:**

$$\boxed{\begin{aligned}\partial_t f_\mathrm{i} + \mathbf{v}\cdot\nabla_\mathbf{x} f_i + \mathbf{a}_\mathrm{i}\cdot\nabla_\mathbf{v} f_\mathrm{i} &= \widetilde{\mathcal{C}}_\mathrm{ii} + \overleftrightarrow{\mathcal{C}}_\mathrm{ie} := \mathcal{C}_\mathrm{i}\\ \partial_t f_\mathrm{e} + \mathbf{v}\cdot\nabla_\mathbf{x} f_e + \mathbf{a}_\mathrm{e}\cdot\nabla_\mathbf{v} f_\mathrm{e} &= \widetilde{\mathcal{C}}_\mathrm{ee} + \overleftrightarrow{\mathcal{C}}_\mathrm{ei} := \mathcal{C}_\mathrm{e}\end{aligned}}$$

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0,$$

$$\partial_t \mathbf{E} - c^2\nabla\times\mathbf{B} = -\mathbf{J}/\epsilon_0,$$

$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\cdot\mathbf{E} = \sigma/\epsilon_0,$$

$$\sigma := \sum_\mathrm{s}\frac{q_\mathrm{s}}{m_\mathrm{s}}\int f_\mathrm{s}\, d\mathbf{v},$$

$$\mathbf{J} := \sum_\mathrm{s}\frac{q_\mathrm{s}}{m_\mathrm{s}}\int \mathbf{v} f_\mathrm{s}\, d\mathbf{v}$$

- **Lorentz force law**

$$\mathbf{a}_\mathrm{i} = \frac{q_i}{m_i}(\mathbf{E} + \mathbf{v}\times\mathbf{B}),$$

$$\mathbf{a}_\mathrm{e} = \frac{q_e}{m_e}(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

- **Collision operator**

Conservation dictates:

$$\int_\mathbf{v}\mathbf{m}\widetilde{\mathcal{C}}_\mathrm{ii} = 0 = \int_\mathbf{v}\mathbf{m}\widetilde{\mathcal{C}}_\mathrm{ee},$$

$$\int_\mathbf{v}\mathbf{m}(\mathcal{C}_\mathrm{i} + \mathcal{C}_\mathrm{e}) = 0$$

where $\mathbf{m} = (1, \mathbf{v}, \|\mathbf{v}\|^2)$.

The source term $\mathcal{C}_\mathrm{s} = \widetilde{\mathcal{C}}_\mathrm{ss} + \overleftrightarrow{\mathcal{C}}_\mathrm{sp}$ is specified by physics, but there is some freedom in how to allocate $\mathcal{C}_\mathrm{s}$ among $\widetilde{\mathcal{C}}_\mathrm{ss}$ and $\overleftrightarrow{\mathcal{C}}_\mathrm{sp}$. For weakly collisional plasma $\int_\mathbf{v}\mathbf{m}\mathcal{C}_\mathrm{s} \approx 0$, and $\widetilde{\mathcal{C}}_\mathrm{ss}$ can be chosen to dominate $\widetilde{\mathcal{C}}_\mathrm{sp}$.

- **Gaussian-BGK collision operator**

For $\mathcal{C}_\mathrm{ss}$ we obtain relaxation closures with a Gaussian-BGK collision operator which relaxes toward a Gaussian distribution:

$$\widetilde{\mathcal{C}}_\mathrm{ss} = C_{\widetilde{\Theta}} = \frac{f_{\widetilde{\Theta}} - f}{\widetilde{\tau}},$$

where the Gaussian distribution $f_{\widetilde{\Theta}}$ shares physically conserved moments with f and has pseudo-temperature $\widetilde{\Theta}$ equal to an affine (not necessarily convex!) combination of the pseudo-temperature Θ and its isotropization:

$$f_{\widetilde{\Theta}} = \frac{\rho e^{\left(-\mathbf{c}\cdot\widetilde{\Theta}^{-1}\cdot\mathbf{c}/2\right)}}{\sqrt{\det(2\pi\widetilde{\Theta})}},$$

$$\Theta := \langle\mathbf{cc}\rangle = \int \mathbf{cc} f d\mathbf{v} \Big/ \int f d\mathbf{v},$$

$$\widetilde{\Theta} := \overline{\nu}\theta\mathbb{I} + \nu\Theta, \quad (\overline{\nu} + \nu = 1),$$

$$\overline{\nu} := 1/\Pr = \tau/\widetilde{\tau}.$$

Here $\widetilde{\tau}$ is the heat flux relaxation period, τ is the relaxation period of deviatoric pressure, and $C_{\widetilde{\Theta}}$ respects entropy if $\widetilde{\Theta}$ is positive definite (i.e. $0 < \overline{\nu} \le 3/2$). In the limit $\overline{\nu} \searrow 0$ heat flux goes to zero and the solution approximates hyperbolic Gaussian-moment (10-moment) gas dynamics.

Use of a Gaussian-BKG collision operator allows one to tune the viscosity $\mu = p\tau$ and the thermal conductivity $k = \frac{5}{2}\frac{\mu}{m\Pr}$.

Evolution equations

$$\bar{\delta}_t \rho_s = 0$$
$$\rho_s d_t \mathbf{u}_s + \nabla p_s + \nabla \cdot \mathbb{P}_s^\circ = q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \mathbf{R}_s$$
$$\tfrac{3}{2} \bar{\delta}_t p_s + p_s \nabla \cdot \mathbf{u}_s + \mathbb{P}_s^\circ : \nabla \mathbf{u}_s + \nabla \cdot \mathbf{q}_s = Q_s$$

Evolved moments

$$\begin{bmatrix} \rho_s \\ \rho_s \mathbf{u}_s \\ \frac{3}{2} p_s \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2} \|\mathbf{c}_s\|^2 \end{bmatrix} f_s \, d\mathbf{v}$$

Definitions

$$\bar{\delta}_t(\alpha) := \partial_t \alpha + \nabla \cdot (\mathbf{u}_s \alpha)$$
$$\mathbf{c}_s := \mathbf{v} - \mathbf{u}_s$$
$$n_s := \rho_s / m_s$$

Relaxation (diffusive) flux closures:

$$\mathbb{P}_s^\circ = \int (\mathbf{c}_s \mathbf{c}_s - \|\mathbf{c}_s\|^2 \mathbb{I}/3) \, f_s \, d\mathbf{v},$$
$$\mathbf{q}_s = \int \tfrac{1}{2} \mathbf{c}_s \|\mathbf{c}_s\|^2 \, f_s \, d\mathbf{v}$$

Interspecies forcing closures:

$$\begin{bmatrix} \mathbf{R}_s \\ Q_s \end{bmatrix} = \int \begin{bmatrix} \mathbf{v} \\ \frac{1}{2} \|\mathbf{c}_s\|^2 \end{bmatrix} \overleftrightarrow{\mathcal{C}}_s \, d\mathbf{v} \approx 0$$

Evolution equations

$$\bar{\delta}_t \rho_s = 0$$

$$\rho_s d_t \mathbf{u}_s + \nabla \cdot \mathbb{P}_s = q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \mathbf{R}_s$$

$$\bar{\delta}_t \mathbb{P}_s + \mathrm{Sym2}(\mathbb{P}_s \cdot \nabla \mathbf{u}_s) + \nabla \cdot \underline{\underline{\underline{q}}}_s = q_s n_s \,\mathrm{Sym2}(\mathbb{T}_s \times \mathbf{B}) + \mathbb{R}_s + \mathbb{Q}_s$$

Evolved moments

$$\begin{bmatrix} \rho_s \\ \rho_s \mathbf{u}_s \\ \mathbb{P}_s \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \mathbf{c}_s \mathbf{c}_s \end{bmatrix} f_s \, d\mathbf{v}$$

Definitions

$$\bar{\delta}_t(\alpha) := \partial_t \alpha + \nabla \cdot (\mathbf{u}_s \alpha)$$

$$\mathbf{c}_s := \mathbf{v} - \mathbf{u}_s$$

$$\mathrm{Sym2}(A) := A + A^T$$

$$n_s := \rho_s / m_s$$

$$\mathbb{T} := \mathbb{P}/n$$

Relaxation (diffusive) flux closures:

$$\underline{\underline{\underline{q}}}_s = \int \mathbf{c}_s \mathbf{c}_s \mathbf{c}_s \, f_s \, d\mathbf{c}_s$$

Relaxation source term closures:

$$\mathbb{R}_s = \int \mathbf{c}_s \mathbf{c}_s \, C_s \, d\mathbf{v} = -\mathbb{P}_s^{\circ}/\tau,$$

Interspecies forcing closures:

$$\begin{bmatrix} \mathbf{R}_s \\ \mathbb{Q}_s \end{bmatrix} = \int \begin{bmatrix} \mathbf{v} \\ \mathbf{c}_s \mathbf{c}_s \end{bmatrix} \overleftrightarrow{\mathcal{C}}_s \, d\mathbf{v} \approx 0$$

Evolution equations

$$\bar{\delta}_t \rho_s = 0$$
$$\rho_s d_t \mathbf{u}_s + \nabla \cdot \mathbb{P}_s = q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \mathbf{R}_s$$
$$\bar{\delta}_t \mathbb{P}_s + \mathrm{Sym2}(\mathbb{P}_s \cdot \nabla \mathbf{u}_s) + \nabla \cdot \underline{\underline{\underline{q}}}_s = q_s n_s \,\mathrm{Sym2}(\mathbb{T}_s \times \mathbf{B}) + \mathbb{R}_s + \mathbb{Q}_s$$
$$\bar{\delta}_t \mathbf{q}_s + \mathbf{q}_s \cdot \nabla \mathbf{u}_s + \underline{\underline{\underline{q}}}_s : \nabla \mathbf{u}_s + \mathbb{P}_s : \nabla \underline{\underline{\Theta}}_s + \mathbb{P}_s \cdot \nabla \theta_s + \nabla \cdot \underline{\underline{R}}_s = \mathrm{Sym3}(\mathbb{P}_s \mathbb{P}_s)/\rho_s + \frac{q_s}{m_s} \underline{\underline{\underline{q}}}_s \times \mathbf{B} + \widetilde{\mathbf{q}}_{ss,t} + \overleftrightarrow{\mathbf{q}}_{s,t}$$

Evolved moments

$$\begin{bmatrix} \rho_s \\ \rho_s \mathbf{u}_s \\ \mathbb{P}_s \\ \mathbf{q}_s \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \mathbf{c}_s \mathbf{c}_s \\ \frac{1}{2} \mathbf{c}_s \|\mathbf{c}_s\|^2 \end{bmatrix} f_s \, d\mathbf{v}$$

Definitions

$$\bar{\delta}_t(\alpha) := \partial_t \alpha + \nabla \cdot (\mathbf{u}_s \alpha)$$
$$\mathbf{c}_s := \mathbf{v} - \mathbf{u}_s, \qquad \mathrm{Sym2}(A) := A + A^T,$$
$$n_s := \rho_s / m_s, \qquad \mathbb{T} := \mathbb{P}/n,$$
$$\underline{\underline{\Theta}} := \mathbb{P}/\rho, \qquad \theta := \mathrm{tr}\, \underline{\underline{\Theta}}/2,$$

Relaxation source term closures:

$$\begin{bmatrix} \mathbb{R}_s \\ \widetilde{\mathbf{q}}_{ss,t} \end{bmatrix} = \int \begin{bmatrix} \mathbf{c}_s \mathbf{c}_s \\ \frac{1}{2} \mathbf{c}_s \|\mathbf{c}_s\|^2 \end{bmatrix} C_{ss} \, d\mathbf{v} = \frac{-1}{\tau_s} \begin{bmatrix} \mathbb{P}_s^\circ \\ \mathrm{Pr}\, \mathbf{q}_s \end{bmatrix}$$

Hyperbolic flux closures:

$$\begin{bmatrix} \underline{\underline{\underline{q}}}_s \\ \underline{\underline{R}}_s \end{bmatrix} = \int \begin{bmatrix} \mathbf{c}_s \mathbf{c}_s \mathbf{c}_s \\ \mathbf{c}_s \mathbf{c}_s \|\mathbf{c}_s\|^2 \end{bmatrix} f_s(\mathbf{c}_s) \, d\mathbf{c}_s$$

Interspecies forcing closures:

$$\begin{bmatrix} \mathbf{R}_s \\ \mathbb{Q}_s \\ \overleftrightarrow{\mathbf{q}}_{s,t} \end{bmatrix} = \int \begin{bmatrix} \mathbf{v} \\ \mathbf{c}_s \mathbf{c}_s \\ \frac{1}{2} \mathbf{c}_s \|\mathbf{c}_s\|^2 \end{bmatrix} \overleftrightarrow{\mathcal{C}}_s \, d\mathbf{v} \approx 0$$

Models which evolve Maxwell's equations and classical gas dynamics fail to satisfy a relativity principle. Magnetohydrodyamics (MHD) remedies this problem by assuming that the light speed is infinite. Then Maxwell's equations simplify to

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \cancel{c^{-2}\partial_t \mathbf{E}}, \qquad \mu_0 \sigma = 0 + \cancel{c^{-2} \nabla \cdot \mathbf{E}}$$

This system is Galilean-invariant, but its relationship to gas-dynamics is fundamentally different:

variable	MHD	2-fluid-Maxwell
$\mathbf{E}$	supplied by *Ohm's law* (from gas dynamics)	evolved (from **B** and **J**)
$\mathbf{J}$	$\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ (comes from **B**)	$\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e)$ (from gas dynamics)
σ	$\sigma = 0$ (quasineutrality) (gas-dynamic constraint)	$\sigma = e(n_i - n_e)$ (electric field constraint)

MHD: charge neutrality

The assumption of charge neutrality reduces the number of gas-dynamic equations that must be solved:

- ***net* density** evolution

  The density of each species is the same:

  $$n_{\mathrm{i}} = n_{\mathrm{e}} = n$$

- ***net* velocity** evolution

  The species fluid velocities can be inferred from the net current, net velocity, and density:

  $$\mathbf{u}_{\mathrm{i}} = \mathbf{u} + \frac{m_{\mathrm{e}}}{m_{\mathrm{i}} + m_{\mathrm{e}}} \frac{\mathbf{J}}{ne}, \qquad \mathbf{u}_{\mathrm{e}} = \mathbf{u} - \frac{m_{\mathrm{i}}}{m_{\mathrm{i}} + m_{\mathrm{e}}} \frac{\mathbf{J}}{ne}.$$

MHD: Ohm's law

For each species $\mathrm{s} \in \{\mathrm{i}, \mathrm{e}\}$, rescaling momentum evolution by $\mathbf{q}_\mathrm{s}/m_\mathrm{s}$ gives the current evolution equation

$$\partial_t \mathbf{J}_\mathrm{s} + \nabla \cdot (\mathbf{u}_\mathrm{s}\mathbf{J}_\mathrm{s} + (q_\mathrm{s}/m_\mathrm{s})\mathbb{P}_\mathrm{s}) = (q_\mathrm{s}^2/m_\mathrm{s})n(\mathbf{E} + \mathbf{u}_\mathrm{s} \times \mathbf{B}) + (q_\mathrm{s}/m_\mathrm{s})\mathbf{R}_\mathrm{s}.$$

Summing over both species and using charge neutrality gives net current evolution:

$$\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) + e\nabla \cdot \left(\frac{\mathbb{P}_\mathrm{i}}{\mathbf{m}_\mathrm{i}} - \frac{\mathbb{P}_\mathrm{e}}{\mathbf{m}_\mathrm{e}} \right) = \frac{e^2\rho}{m_\mathrm{i} m_\mathrm{e}} \left(\mathbf{E} + \left(\mathbf{u} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \right) \times \mathbf{B} - \frac{\mathbf{R}_\mathrm{e}}{en} \right).$$

A closure for the collisional term is $\frac{\mathbf{R}_\mathrm{e}}{en} = \boldsymbol{\eta} \cdot \mathbf{J} + \boldsymbol{\beta}_\mathrm{e} \cdot \mathbf{q}_\mathrm{e}$.

Ohm's law is current evolution solved for the electric field:

$$\boxed{\begin{aligned}
\mathbf{E} =& \mathbf{B} \times \mathbf{u} && \text{(ideal term)} \\
&+ \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B} && \text{(Hall term)} \\
&+ \boldsymbol{\eta} \cdot \mathbf{J} && \text{(resistive term)} \\
&+ \boldsymbol{\beta}_\mathrm{e} \cdot \mathbf{q}_\mathrm{e} && \text{(thermoelectric term)} \\
&+ \frac{1}{e\rho} \nabla \cdot (m_\mathrm{e}\mathbb{P}_\mathrm{i} - m_\mathrm{i}\mathbb{P}_\mathrm{e}) && \text{(pressure term)} \\
&+ \frac{m_\mathrm{i} m_\mathrm{e}}{e^2\rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right] && \text{(inertial term).}
\end{aligned}}$$

Ohm's law gives an implicit closure to the induction equation, $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$ (so retaining the inertial term entails an implicit numerical method).

Equations of 5-moment 2-fluid MHD

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

Ohm's law

$$\mathbf{E} = \frac{\mathbf{R}_\mathrm{e}}{en} + \mathbf{B} \times \mathbf{u} + \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B}$$
$$+ \frac{1}{e\rho} \nabla \cdot (m_\mathrm{e}(p_\mathrm{i}\mathbb{I} + \mathbb{P}_\mathrm{i}^\circ) - m_\mathrm{i}(p_\mathrm{e}\mathbb{I} + \mathbb{P}_\mathrm{e}^\circ))$$
$$+ \frac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right]$$

Pressure evolution

$$\tfrac{3}{2} n d_t T_\mathrm{i} + p_\mathrm{i} \nabla \cdot \mathbf{u}_\mathrm{i} + \mathbb{P}^\circ{}_\mathrm{i} : \nabla \mathbf{u}_\mathrm{i} + \nabla \cdot \mathbf{q}_\mathrm{i} = Q_\mathrm{i},$$

$$\tfrac{3}{2} n d_t T_\mathrm{e} + p_\mathrm{e} \nabla \cdot \mathbf{u}_\mathrm{e} + \mathbb{P}^\circ{}_\mathrm{e} : \nabla \mathbf{u}_\mathrm{e} + \nabla \cdot \mathbf{q}_\mathrm{e} = Q_\mathrm{e};$$

Definitions:

$$d_t := \partial_t + \mathbf{u}_\mathrm{s} \cdot \nabla,$$

$$\mathbb{P}^\mathrm{d} := \rho_i \mathbf{w}_\mathrm{i} \mathbf{w}_\mathrm{i} + \rho_e \mathbf{w}_\mathrm{e} \mathbf{w}_e = m_\mathrm{red} n \mathbf{ww}$$

$$\mathbf{w} = \frac{\mathbf{J}}{en}, \quad \mathbf{w}_\mathrm{i} = \frac{m_\mathrm{e}}{m_\mathrm{tot}} \mathbf{w}, \quad \mathbf{w}_\mathrm{e} = \frac{-m_\mathrm{i}}{m_\mathrm{tot}} \mathbf{w},$$

Closures:

$$\mathbb{P}_\mathrm{s}^\circ = -2\boldsymbol{\mu} : (\nabla \mathbf{u})^\circ$$

$$\mathbf{q}_\mathrm{s} = -\mathbf{k} \cdot \nabla T$$

$$\frac{\mathbf{R}_\mathrm{e}}{en} = \boldsymbol{\eta} \cdot \mathbf{J} + \boldsymbol{\beta}_\mathrm{e} \cdot \mathbf{q}_\mathrm{e}$$

$$Q_\mathrm{s} = ?$$

mass and momentum:

$$\partial_t \rho + \nabla\cdot(\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla\cdot(\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J}\times\mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0,$$

$$\mathbf{J} = \mu_0^{-1}\nabla\times\mathbf{B}$$

Ohm's law

$$\mathbf{E} = \frac{\mathbf{R}_\mathrm{e}}{en} + \mathbf{B}\times\mathbf{u} + \frac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{J}\times\mathbf{B} + \frac{1}{e\rho}\nabla\cdot(m_\mathrm{e}\mathbb{P}_\mathrm{i} - m_\mathrm{i}\mathbb{P}_\mathrm{e}) + \frac{m_\mathrm{i}m_\mathrm{e}}{e^2\rho}\left[\partial_t\mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{JJ}\right)\right]$$

Definitions:

$$d_t := \partial_t + \mathbf{u}_\mathrm{s}\cdot\nabla,$$

$$\mathbb{P}^\mathrm{d} := \rho_i\mathbf{w}_\mathrm{i}\mathbf{w}_\mathrm{i} + \rho_e\mathbf{w}_\mathrm{e}\mathbf{w}_e$$

$$\mathbf{w}_\mathrm{i} = \frac{m_\mathrm{e}\mathbf{J}}{e\rho}, \quad \mathbf{w}_\mathrm{e} = -\frac{m_\mathrm{i}\mathbf{J}}{e\rho}$$

Closures:

$$\mathbb{R}_\mathrm{s} = -\tfrac{1}{\tau}\mathbb{P}_\mathrm{s}^\circ$$

$$\underline{\underline{\underline{q}}}_\mathrm{s} = -\tfrac{2}{5}\mathbf{K}_\mathrm{s} \,\vdots\, \mathrm{Sym3}\left(\frac{\mathbb{T}_\mathrm{s}}{T_\mathrm{s}}\cdot\nabla\mathbb{T}_\mathrm{s}\right)$$

$$\frac{\mathbf{R}_\mathrm{e}}{en} = \boldsymbol{\eta}\cdot\mathbf{J} + \boldsymbol{\beta}_\mathrm{e}\cdot\mathbf{q}_\mathrm{e}$$

$$\mathbb{Q}_\mathrm{s} = ?$$

Pressure evolution

$$n_\mathrm{i} d_t\mathbb{T}_\mathrm{i} + \mathrm{Sym2}(\mathbb{P}_\mathrm{i}\cdot\nabla\mathbf{u}_\mathrm{i}) + \nabla\cdot\underline{\underline{\underline{q}}}_\mathrm{i} = \frac{q_\mathrm{i}}{m_\mathrm{i}}\mathrm{Sym2}(\mathbb{P}_\mathrm{i}\times\mathbf{B}) + \mathbb{R}_\mathrm{i} + \mathbb{Q}_\mathrm{i},$$

$$n_\mathrm{e} d_t\mathbb{T}_\mathrm{e} + \mathrm{Sym2}(\mathbb{P}_\mathrm{e}\cdot\nabla\mathbf{u}_\mathrm{e}) + \nabla\cdot\underline{\underline{\underline{q}}}_\mathrm{e} = \frac{q_\mathrm{e}}{m_\mathrm{e}}\mathrm{Sym2}(\mathbb{P}_\mathrm{e}\times\mathbf{B}) + \mathbb{R}_\mathrm{e} + \mathbb{Q}_\mathrm{e}$$

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

Ohm's law

$$\mathbf{E} = \frac{\mathbf{R}_\mathrm{e}}{en} + \mathbf{B} \times \mathbf{u} + \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B} + \frac{1}{e\rho} \nabla \cdot (m_\mathrm{e}\mathbb{P}_\mathrm{i} - m_\mathrm{i}\mathbb{P}_\mathrm{e}) + \frac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho}\left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ}\right)\right]$$

Diffusive relaxation closures:

$$\frac{\mathbf{R}_\mathrm{e}}{en} = \boldsymbol{\eta} \cdot \mathbf{J} + \boldsymbol{\beta}_\mathrm{e} \cdot \mathbf{q}_\mathrm{e}$$

Relaxation source term closures:

$$\mathbb{R}_\mathrm{s} = -\mathbb{P}_\mathrm{s}^\circ / \tau_\mathrm{s}$$

$$\widetilde{\mathbf{q}}_{\mathrm{ss},t} = -\mathbf{q}_\mathrm{s} / \widetilde{\tau}_\mathrm{s}$$

Hyperbolic flux closures:

$$\begin{bmatrix} \underline{\underline{\underline{q}}}_\mathrm{s} \\ \underline{\underline{R}}_\mathrm{s} \end{bmatrix} = \int \begin{bmatrix} \mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s} \\ \mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s}\|\mathbf{c}_\mathrm{s}\|^2 \end{bmatrix} f_\mathrm{s}(\mathbf{c}_\mathrm{s})\, d\mathbf{c}_\mathrm{s}$$

Interspecies forcing closures:

$$\begin{bmatrix} \mathbf{R}_\mathrm{s} \\ \mathbb{Q}_\mathrm{s} \\ \overleftrightarrow{\mathbf{q}}_{\mathrm{s},t} \end{bmatrix} = ?$$

Pressure evolution

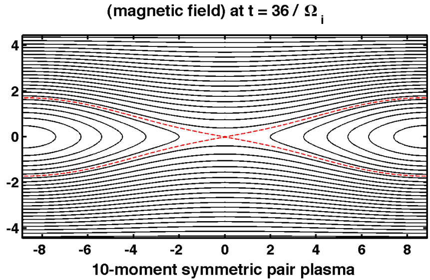
$$n_\mathrm{i} d_t \mathbb{T}_\mathrm{i} + \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \cdot \nabla \mathbf{u}_\mathrm{i}) + \nabla \cdot \underline{\underline{\underline{q}}}_\mathrm{i} = \frac{q_\mathrm{i}}{m_\mathrm{i}} \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \times \mathbf{B}) + \mathbb{R}_\mathrm{i} + \mathbb{Q}_\mathrm{i},$$

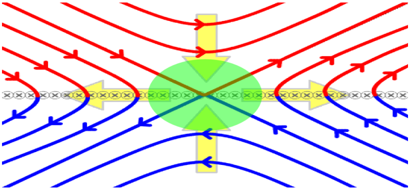
$$n_\mathrm{e} d_t \mathbb{T}_\mathrm{e} + \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \cdot \nabla \mathbf{u}_\mathrm{e}) + \nabla \cdot \underline{\underline{\underline{q}}}_\mathrm{e} = \frac{q_\mathrm{e}}{m_\mathrm{e}} \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \times \mathbf{B}) + \mathbb{R}_\mathrm{e} + \mathbb{Q}_\mathrm{e}$$

Heat flux evolution

$$\bar{\delta}_t \mathbf{q}_\mathrm{s} + \mathbf{q}_\mathrm{s} \cdot \nabla \mathbf{u}_\mathrm{s} + \underline{\underline{\underline{q}}}_\mathrm{s} : \nabla \mathbf{u}_\mathrm{s} + \mathbb{P}_\mathrm{s} : \nabla \underline{\underline{\Theta}}_\mathrm{s} + \tfrac{3}{2} \mathbb{P}_\mathrm{s} \cdot \nabla \theta_\mathrm{s} + \tfrac{1}{2} \nabla \cdot \underline{\underline{R}}_\mathrm{s} = \rho(3\theta\Theta + 2\Theta \cdot \Theta) + \frac{q_\mathrm{s}}{m_\mathrm{s}} \underline{\underline{\underline{q}}}_\mathrm{s} \times \mathbf{B} + \widetilde{\mathbf{q}}_{\mathrm{ss},t} + \overleftrightarrow{\mathbf{q}}_{\mathrm{s},t}$$

Define a ***symmetric 2D*** problem to be a 2D problem symmetric under 180-degree rotation about the origin (0). In our simulations of symmetric 2D reconnection the origin is an X-point of the magnetic field:





The following slides identify requirements for fast magnetic reconnection by analyzing the solution near the X-point. We argue that, for accurate resolution of the electron pressure tensor near the X-point, a fluid model of fast reconnection (1) must resolve two-fluid effects, (2) should resolve strong pressure anisotropy, and (3) must admit viscosity and heat flow.

All equations in this part assume a steady-state solution to a symmetric 2D problem and are evaluated at the origin (0).

1. Ohm's law: fast reconnection needs two-fluid effects.

Ohm's law is net electrical current evolution solved for the electric field. Assuming symmetry across the X-point, the steady-state Ohm's law evaluated at the X-point reads

$$\mathbf{E}^{\|} = \frac{\mathbf{R}_{\mathrm{e}}^{\|}}{en} + \frac{1}{e\rho}\left[\nabla\cdot(m_{\mathrm{e}}\mathbb{P}_{\mathrm{i}} - m_{\mathrm{i}}\mathbb{P}_{\mathrm{e}})\right]^{\|} \qquad \text{at 0 for } \partial_t = 0.$$

Fast reconnection is nearly collisionless, so the collisional drag term $\mathbf{R}_{\mathrm{e}}$ should be negligible.

For *pair plasma*, the pressure term is zero unless the pressure tensors of the two species are allowed to differ. In fact, kinetic simulations of collisionless antiparallel reconnection admit fast rates of reconnection [BeBh07], and we get similar rates using a two-fluid Gaussian-moment model of pair plasma with pressure isotropization [Jo11].

For *hydrogen plasma*, the electron pressure term chiefly supports reconnection, and the Hall term $\frac{m_{\mathrm{i}}-m_{\mathrm{e}}}{e\rho}\mathbf{J}\times\mathbf{B}$, although zero at the X-point, appears to accelerate the rate of reconnection [ShDrRoDe01].

2. Pressure anisotropy at X-point needs an extended-moment model.

For antiparallel reconnection, the pressure tensor becomes strongly agyrotropic in the immediate vicinity of the X-point [Br11, ScGr06]. Stress closures for the Maxwellian-moment model assume that the pressure tensor is nearly isotropic. In contrast, the assumptions of the Gaussian-moment model (that the distribution of particle velocities is nearly Gaussian) can hold even for strongly anisotropic pressure. In practice, we have found good agreement of the Gaussian-moment two-fluid model with kinetic simulations [Jo11, JoRo10]:

- Reconnection rates are approximately correct.
- Reconnection is primarily supported by pressure agyrotropy.
- There is qualitatively good resolution of the electron pressure tensor near the X-point even when the pressure becomes strongly agyrotropic.

3. Theory: steady collisionless reconnection requires viscosity & heat flux

For a symmetric 2D problem, the origin is a stagnation point. Informally, we show that steady reconnection is not possible without heat production near the stagnation point and that a mechanism for heat flow is therefore necessary to prevent a heating singularity at the stagnation point. Formally, define a solution to be **nonsingular** if density and pressure are finite, strictly positive, and smooth; we show that a steady-state solution to a symmetric 2D problem must be singular if viscosity or heat flux is absent.

3a. Steady collisionless reconnection requires viscosity.

By *Faraday's law* the rate of reconnection is $\mathbf{E}^{\parallel}(0)$ (the out-of-plane electric field evaluated at the origin). Momentum evolution implies

$$\mathbf{E}^{\parallel}(0) = \frac{-\mathbf{R}_{\mathrm{s}}^{\parallel}}{\sigma_{\mathrm{s}}} + \frac{(\nabla \cdot \mathbb{P}_{\mathrm{s}})^{\parallel}}{\sigma_{\mathrm{s}}} \qquad \text{at 0 for } \partial_t = 0, \tag{1}$$

where σ_{s} is charge density. For collisionless reconnection the drag force $\mathbf{R}_{\mathrm{s}}$ should be negligible. If the pressure is isotropic or gyrotropic in a neighborhood of 0, then $\nabla \cdot \mathbb{P}_{\mathrm{s}}$ is zero. That is, inviscid models do not admit steady reconnection [HeKuBi04].

3b. Theorem: Steady collisionless reconnection requires heat flux.

Viscous models generate heat near the X-point. Symmetry implies that the X-point is a stagnation point. An adiabatic fluid model provides no mechanism for heat to dissipate away from the X-point. As a result, viscous adiabatic models develop a temperature singularity near the X-point when used to simulate sustained reconnection. Numerically, when we simulated the GEM magnetic reconnection challenge problem using an adiabatic Gaussian-moment model with pressure isotropization (viscosity), shortly after the peak reconnection rate temperature singularities developed near the X-point. Theoretically, we have the following steady-state result:

Theorem [Jo11]. *For a 2D problem invariant under 180-degree rotation about 0 (the origin), steady-state nonsingular magnetic reconnection is impossible without heat flux for a Maxwellian-moment or Gaussian-moment model that uses linear (gyrotropic) closure relations that satisfy a positive-definiteness condition and respect entropy (in the Maxwellian limit).*

Proof (Maxwellian case)

Let $'$ denote a partial derivative (∂_x or ∂_y) evaluated at 0. Conservation of mass and pressure evolution imply the **entropy evolution equation**:

$$p_\mathrm{s}\mathbf{u}_\mathrm{s} \cdot \nabla s = 2\boldsymbol{e}_\mathrm{s}^\circ : \boldsymbol{\mu}_\mathrm{s} : \boldsymbol{e}_\mathrm{s}^\circ - \nabla \cdot \mathbf{q}_\mathrm{s} + Q_\mathrm{s}, \tag{2}$$

where $\boldsymbol{e}_\mathrm{s}^\circ = \nabla \mathbf{u}_\mathrm{s}^\circ$ is deviatoric strain, $-\mathbb{P}_\mathrm{s}^\circ = 2\boldsymbol{\mu}_\mathrm{s} : \boldsymbol{e}_\mathrm{s}^\circ$ is deviatoric stress, and $\boldsymbol{\mu}_\mathrm{s}$ is the viscosity tensor. Assume that $\mathbf{q}_\mathrm{s} = 0$ near 0. Evaluating equation (2) at 0 and invoking symmetries yields

$$\boldsymbol{e}_\mathrm{s}^\circ : \boldsymbol{\mu}_\mathrm{s} : \boldsymbol{e}_\mathrm{s}^\circ = -Q_\mathrm{s}.$$

Assume that $\boldsymbol{\mu}$ is positive-definite. Assume that thermal heat exchange conserves energy: $Q_\mathrm{i} + Q_\mathrm{e} = 0$. So Q_s must be zero, so $\boldsymbol{e}_\mathrm{s}^\circ = 0$ at 0. Evaluating the second derivative of equation (2) at 0 and invoking symmetries yields $(\boldsymbol{e}_\mathrm{s}^\circ)' : \boldsymbol{\mu} : (\boldsymbol{e}_\mathrm{s}^\circ)' = -Q_\mathrm{s}''$, which by conservation of energy ($Q_\mathrm{i}'' + Q_\mathrm{e}'' = 0$) must be nonpositive for one of the two species (which we take to be s) for differentiation along two orthogonal directions. Using that $\boldsymbol{\mu}$ is positive-definite, $(\boldsymbol{e}_\mathrm{s}^\circ)' = 0$. Therefore, $-(\mathbb{P}_\mathrm{s}^\circ)' = 2(\boldsymbol{\mu}_\mathrm{s} : \boldsymbol{e}_\mathrm{s}^\circ)' = 0$. Since this relation holds for two orthogonal directions, $\nabla \mathbb{P}_\mathrm{s} = 0$ at 0, so $\nabla \cdot \mathbb{P}_\mathrm{s} = 0$ at 0. So equation (1) says that $\mathbf{E}^\parallel(0) = 0$, i.e., there is no reconnection. $\square$

Proof (Gaussian case)

Let $'$ denote a partial derivative (∂_x or ∂_y) evaluated at 0. Conservation of mass and pressure evolution imply the **entropy evolution equation**:

$$n_s \mathbf{u}_s \cdot \nabla s = -2\tau^{-1}\mathbb{P}_s^{-1} : \mathbf{C} : \mathbb{P}_s^\circ - \mathbb{P}_s^{-1} : \nabla \cdot \mathbf{q}_s + \mathbb{P}_s^{-1} : \mathbb{Q}_s, \tag{3}$$

where $\mathbb{R}_s := \tau^{-1}\mathbf{C} : \mathbb{P}^\circ$ is traceless. Assume that $\underline{\underline{\underline{q}}}_s = 0$ near 0. Evaluating equation (3) at 0 and invoking symmetries yields

$$0 = -2\tau^{-1}(\mathbb{P}_s^{-1}) : \mathbf{C} : (\mathbb{P}_s^\circ) + \mathbb{P}_s^{-1} : \mathbb{Q}_s. \tag{4}$$

Assume that $\mathbf{C}$ satisfies the positive-definiteness criterion $-(\mathbb{P}_s^{-1}) : \mathbf{C} : (\mathbb{P}_s^\circ) \geq 0$. Assume that a linear closure is used for $\mathbb{Q}_i$ and $\mathbb{Q}_s$ (thermal heat exchange) in terms of $\mathbb{P}_i$ and $\mathbb{P}_e$ which respects total gas-dynamic entropy at 0. Then $\mathbb{P}_s^\circ = 0$ at 0. Evaluating the second derivative of equation (3) at 0 and invoking symmetries yields

$$0 = -2\tau^{-1}(\mathbb{P}_s^{-1})' : \mathbf{C} : (\mathbb{P}_s^\circ)' + (\mathbb{P}_s^{-1} : \mathbb{Q}_s)''. \tag{5}$$

Using that $\mathbf{C}$ is positive-definite, $(\mathbb{P}_s^\circ)' = 0$ for a species s. That is, $\nabla \mathbb{P}_s = 0$ at 0, so $\nabla \cdot \mathbb{P}_s = 0$ at 0. So equation (1) says that $\mathbf{E}^\parallel(0) = 0$, i.e., there is no reconnection. □

In this second half we present, as the simplest model satisfying these requirements, a Gaussian-BGK closure of Gaussian-moment two-fluid MHD. A Gaussian-BGK collision operator relaxes the particle velocity distribution toward a Gaussian distribution. We assume a Gaussian-BGK collision operator and use a Chapman-Enskog expansion to derive a closure for Maxwellian-moment and Gaussian-moment MHD.

Magnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

Ohm's law:

$$\mathbf{E} = \boldsymbol{\eta} \cdot \mathbf{J} + \mathbf{B} \times \mathbf{u} + \frac{m_i - m_e}{e\rho} \mathbf{J} \times \mathbf{B}$$
$$+ \frac{1}{e\rho} \nabla \cdot (m_e \mathbb{P}_i - m_i \mathbb{P}_e)$$
$$+ \frac{m_i m_e}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_i - m_e}{e\rho} \mathbf{JJ} \right) \right]$$

Mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$
$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_i + \mathbb{P}_e + \mathbb{P}^d) = \mathbf{J} \times \mathbf{B}$$

Pressure evolution:

$$\boxed{\begin{aligned} \tfrac{3}{2} n d_t T_i + p_i \nabla \cdot \mathbf{u}_i + \mathbb{P}_i^\circ : \nabla \mathbf{u}_i + \nabla \cdot \mathbf{q}_i &= Q_i \\ \tfrac{3}{2} n d_t T_e + p_e \nabla \cdot \mathbf{u}_e + \mathbb{P}_e^\circ : \nabla \mathbf{u}_e + \nabla \cdot \mathbf{q}_e &= Q_e \end{aligned}}$$

Closures:

$$\boxed{\begin{aligned} \mathbb{P}_s^\circ &= -2 \boldsymbol{\mu}_s : \boldsymbol{e}_s^\circ \\ \mathbf{q}_s &= -\mathbf{k}_s \cdot \nabla T_s \\ (Q_s &= Q_s^f + Q_s^t) \end{aligned}}$$

Definitions:

$$\begin{aligned} d_t &= \partial_t + \mathbf{u}_s \cdot \nabla \\ \mathbf{J} &= \mu_0^{-1} \nabla \times \mathbf{B} \\ \boldsymbol{e}_s^\circ &= (\nabla \mathbf{u}_s)^\circ \\ \rho &= (m_i + m_e) n \\ p_s &= n T_s \\ \mathbb{P}_s &= p_s \mathbb{I} + \mathbb{P}_s^\circ \\ \mathbb{P}^d &= \rho_i \mathbf{w}_i \mathbf{w}_i + \rho_e \mathbf{w}_e \mathbf{w}_e \end{aligned}$$

$$\mathbf{w}_i = \frac{m_e \mathbf{J}}{e\rho}, \quad \mathbf{w}_e = -\frac{m_i \mathbf{J}}{e\rho}$$

The Gaussian-moment model evolves full pressure tensors rather than scalar pressure; the equations are identical to those of Maxwellian-moment two-fluid MHD except for the following.

Pressure tensor evolution

$$
\boxed{\begin{aligned}
n d_t \mathbb{T}_\mathrm{i} + \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \cdot \nabla \mathbf{u}_\mathrm{i}) + \nabla \cdot \underline{\underline{q}}_\mathrm{i} &= \frac{q_\mathrm{i}}{m_\mathrm{i}} \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \times \mathbf{B}) + \mathbb{R}_\mathrm{i} + \mathbb{Q}_\mathrm{i} \\
n d_t \mathbb{T}_\mathrm{e} + \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \cdot \nabla \mathbf{u}_\mathrm{e}) + \nabla \cdot \underline{\underline{q}}_\mathrm{e} &= \frac{q_\mathrm{e}}{m_\mathrm{e}} \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \times \mathbf{B}) + \mathbb{R}_\mathrm{e} + \mathbb{Q}_\mathrm{e}
\end{aligned}}
$$

Closures:

$$
\boxed{\begin{aligned}
\mathbb{R}_\mathrm{s} &= -\mathbb{P}^\circ_\mathrm{s} / \tau_\mathrm{s} \\
\underline{\underline{q}}_\mathrm{s} &= -\tfrac{2}{5} \mathbf{K}_\mathrm{s} \,\vdots\, \mathrm{Sym3}\left(\pi \cdot \nabla \mathbb{T}_\mathrm{s}\right) \\
(\mathbb{Q}_\mathrm{s} &= \mathbb{Q}^\mathrm{f}_\mathrm{s} + \mathbb{Q}^\mathrm{t}_\mathrm{s})
\end{aligned}}
$$

Definitions:

$$
\pi = \frac{\mathbb{P}}{p} = \frac{\mathbb{T}}{T}
$$

$$
\mathrm{Sym2} = X \mapsto X + X^\mathrm{T}
$$

$$
\mathrm{Sym3} = \left\{ \begin{matrix} \text{thrice symmetric part} \\ \text{of third-order tensor} \end{matrix} \right\}
$$

Assuming a Gaussian-BGK intraspecies collision operator and performing a Chapman-Enskog expansion about an assumed distribution yields closures for deviatoric pressure and heat flux.

For the Maxwell-moment model we expand about a Maxwellian distribution and obtain implicit closures for heat flux and deviatoric pressure [Woods04]:

$$\mathbf{q} + \widetilde{\varpi}\mathbf{b} \times \mathbf{q} = -k\nabla T, \tag{6}$$

$$\mathbb{P}^\circ + \mathrm{Sym2}(\varpi\mathbf{b} \times \mathbb{P}^\circ) = -\mu 2\boldsymbol{e}^\circ, \tag{7}$$

where μ is viscosity, k is heat conductivity, $\varpi := \tau\omega_c$ is the gyrofrequency per momentum diffusion rate, $\widetilde{\varpi} := \varpi/\,\mathrm{Pr}$ is the gyrofrequency per thermal diffusion rate, and Pr is the *Prandtl number;* the gyrofrequency is $\omega_c := q|\mathbf{B}|/m$, and $\mathbf{b} := \mathbf{B}/|\mathbf{B}|$.

For the Gaussian-moment model we expand about a Gaussian distribution and obtain the relaxation closure $\mathbb{R}_\mathrm{s} = -\mathbb{P}^\circ_\mathrm{s}/\tau_\mathrm{s}$ and an implicit closure relation for the heat flux tensor [Jo11, McGr08]:

$$\boxed{\underline{\underline{\underline{q}}} + \mathrm{Sym3}(\widetilde{\varpi}\mathbf{b} \times \underline{\underline{\underline{q}}}) = -\tfrac{2}{5}k\,\mathrm{Sym3}\,(\boldsymbol{\pi} \cdot \nabla\mathbb{T})}. \tag{8}$$

In this frame the species index s *is suppressed. All products of even-order tensors are* ***splice products*** *satisfying*

$$
\begin{aligned}
(AB)_{j_1 j_2 k_1 k_2} &:= A_{j_1 k_1} B_{j_2 k_2}, \\
(ABC)_{j_1 j_2 j_3 k_1 k_2 k_3} &:= A_{j_1 k_1} B_{j_2 k_2} C_{j_3 k_3},
\end{aligned}
$$

Definitions:

$$
\begin{aligned}
\mathbb{I}_{\|} &:= \mathbf{bb}, \\
\mathbb{I}_{\perp} &:= \mathbb{I} - \mathbf{bb}, \\
\mathbb{I}_{\wedge} &:= \mathbf{b} \times \mathbb{I}.
\end{aligned}
$$

Solving equations (6–7) for $\mathbf{q}$ and $\mathbb{P}^\circ$ gives

$$
\begin{aligned}
\mathbf{q} &= -k\widetilde{\mathbf{k}} \cdot \nabla T, \\
\mathbb{P}^\circ &= -\,\mathrm{Sym2}(\mu\widetilde{\boldsymbol{\mu}} : \boldsymbol{e}^\circ),
\end{aligned}
$$

where [Woods04]

$$
\begin{aligned}
\widetilde{\mathbf{k}} &= \mathbb{I}_{\|} + \tfrac{1}{1+\widetilde{\varpi}^2}(\mathbb{I}_{\perp} - \widetilde{\varpi}\mathbb{I}_{\wedge}), \\
\widetilde{\boldsymbol{\mu}} &= \tfrac{1}{2}(3\mathbb{I}_{\|}^2 + \mathbb{I}_{\perp}^2) + \tfrac{2}{1+\varpi^2}(\mathbb{I}_{\perp}\mathbb{I}_{\|} - \varpi\mathbb{I}_{\wedge}\mathbb{I}_{\|}) \\
&\quad + \tfrac{1}{1+4\varpi^2}(\tfrac{1}{2}(\mathbb{I}_{\perp}^2 - \mathbb{I}_{\wedge}^2) - 2\varpi\mathbb{I}_{\wedge}\mathbb{I}_{\perp}).
\end{aligned}
$$

Solving equation (8) for $\underset{\equiv}{q}$ gives [Jo11]

$$
\begin{aligned}
\underset{\equiv}{q} &= -\,\mathrm{Sym}(\tfrac{2}{5}k\widetilde{\mathbf{K}} \,\vdots\, \mathrm{Sym3}(\boldsymbol{\pi}\cdot\nabla\mathbb{T})), \\
\widetilde{\mathbf{K}} &= \left(\mathbb{I}_{\|}^3 + \tfrac{3}{2}\mathbb{I}_{\|}(\mathbb{I}_{\perp}^2 + \mathbb{I}_{\wedge}^2)\right) \\
&\quad + \tfrac{3}{1+\widetilde{\varpi}^2}\left(\mathbb{I}_{\perp}\mathbb{I}_{\|}^2 - \widetilde{\varpi}\mathbb{I}_{\wedge}\mathbb{I}_{\|}^2\right) \\
&\quad + \tfrac{3}{1+4\widetilde{\varpi}^2}\left(\tfrac{1}{2}(\mathbb{I}_{\perp}^2 - \mathbb{I}_{\wedge}^2)\mathbb{I}_{\|} - 2\widetilde{\varpi}\mathbb{I}_{\wedge}\mathbb{I}_{\perp}\mathbb{I}_{\|}\right) \\
&\quad + (k_0\mathbb{I}_{\perp}^3 + k_1\mathbb{I}_{\wedge}\mathbb{I}_{\perp}^2 + k_2\mathbb{I}_{\wedge}^2\mathbb{I}_{\perp} + k_3\mathbb{I}_{\wedge}^3),
\end{aligned}
$$

where

$$
\begin{aligned}
k_3 &:= \frac{-6\widetilde{\varpi}^3}{1 + 10\widetilde{\varpi}^2 + 9\widetilde{\varpi}^4} &&= -(2/3)\widetilde{\varpi}^{-1} + \mathcal{O}(\widetilde{\varpi}^{-3}), \\
k_2 &:= \frac{6\widetilde{\varpi}^2 + 3\widetilde{\varpi}(1 + 3\widetilde{\varpi}^2)k_3}{1 + 7\widetilde{\varpi}^2} &&= \mathcal{O}(\widetilde{\varpi}^{-2}), \\
k_1 &:= \frac{-3\widetilde{\varpi} + 2\widetilde{\varpi}k_2}{1 + 3\widetilde{\varpi}^2} &&= -\widetilde{\varpi}^{-1} + \mathcal{O}(\widetilde{\varpi}^{-3}), \\
k_0 &:= 1 + \widetilde{\varpi}k_1 &&= \mathcal{O}(\widetilde{\varpi}^{-2}).
\end{aligned}
$$

For collisionless reconnection the interspecies collisional terms should not be necessary for fast reconnection and should be small in comparison to the intraspecies collisional terms. Nevertheless, for completeness we give a linear relaxation closure.

For **thermal equilibration** one can relax toward the average temperature

$$Q^{\mathrm{t}}_{\mathrm{s}} = \tfrac{3}{2} K\, n^2 (T_0 - T_s),$$

where $2T_0 := T_i + T_e$, or toward an average temperature tensor

$$\mathbb{Q}^{\mathrm{t}}_{\mathrm{s}} = K\, n^2 (\mathbb{T}_0 - \mathbb{T}_s),$$

where $2\mathbb{T}_0 := \widetilde{\mathbb{T}}_i + \widetilde{\mathbb{T}}_e$ and

$$\widetilde{\mathbb{T}}_{\mathrm{s}} := \bar{\nu} T_{\mathrm{s}} \mathbb{I} + \nu \mathbb{T}_{\mathrm{s}},$$

where $\bar{\nu} + \nu = 1$, $0 \le \bar{\nu} \le \frac{3}{2}$ and $\bar{\nu}$ might be 1 or Pr^{-1}. Note that the equilibration rate is nK.

Frictional heating results from the interspecies drag force and can be allocated among species in inverse proportion to particle mass:

$$Q^{\mathrm{f}} := Q^{\mathrm{f}}_{\mathrm{i}} + Q^{\mathrm{f}}_{\mathrm{e}} = \boldsymbol{\eta} : \mathbf{JJ}$$
$$m_{\mathrm{i}} Q^{\mathrm{f}}_{\mathrm{i}} = m_{\mathrm{e}} Q^{\mathrm{f}}_{\mathrm{e}}$$

The frictional tensor heating also must be allocated among directions:

$$\mathbb{Q}^{\mathrm{f}} = (\alpha_{\parallel} - \alpha_{\perp})\,\mathrm{Sym2}(\boldsymbol{\eta} \cdot \mathbf{JJ}) + \alpha_{\perp} \boldsymbol{\eta} : \mathbf{JJ}\, \mathbb{I},$$
$$\mathbb{Q}^{\mathrm{f}}_{\mathrm{i}} = \frac{m_{\mathrm{e}}}{m_{\mathrm{e}} + m_{\mathrm{i}}} \mathbb{Q}^{\mathrm{f}},$$
$$\mathbb{Q}^{\mathrm{f}}_{\mathrm{e}} = \frac{m_{\mathrm{i}}}{m_{\mathrm{e}} + m_{\mathrm{i}}} \mathbb{Q}^{\mathrm{f}}.$$

where $\alpha_{\parallel} + 2\alpha_{\perp} = 1$ and $0 \le \alpha_{\parallel} \le 1$.

Diffusion

$$\mu_s = \tau_s n T_s$$
$$\tfrac{2}{5} k_s = \frac{\mu_s}{m_s \, \mathrm{Pr}_s}$$

Relaxation periods

$$\tau_0 := \frac{12\pi^{3/2}}{\ln\Lambda}\left(\frac{\epsilon_0}{e^2}\right)^2$$

$$n\,\tau'_{ss} := \tau_0 \sqrt{m_s \det(\mathbb{T}_s)}$$

(Using $\sqrt{\det(\mathbb{T}_s)}$, not $T^{3/2}$, so that heat flux tensor closure maintains positivity.)

Braginskii

$$\tau_i = .96\tau'_{ii}$$
$$\tau_e = .52\tau'_{ee}$$
$$\mathrm{Pr}_i = .61 \approx \tfrac{2}{3}$$
$$\mathrm{Pr}_e = .58 \approx \tfrac{2}{3}$$

Interspecies (neglectable)

$$K^{-1} := \tau_0 \frac{m_i m_e}{\sqrt{2}} \left(\frac{T'}{m_{\mathrm{red}}}\right)^{3/2}$$
$$2\tau_{ei}^{\epsilon,\mathrm{Br}} = (K\,n)^{-1} \approx \tau_e^{\mathrm{Br}} \frac{m_i}{m_e}$$
$$\eta_0 := \lim_{\varpi\to\infty} \eta_\perp = \frac{m_e}{e^2 n \tau_e^{\mathrm{Br}}}, \qquad (9)$$
$$\eta_\parallel := .51\eta_0,$$

with m_{red} and T' defined by

$$m_{\mathrm{red}}^{-1} := m_i^{-1} + m_e^{-1},$$
$$\frac{T'}{m_{\mathrm{red}}} := \frac{T_i}{m_i} + \frac{T_e}{m_e}$$

Braginskii parameters

$$\tau_i^{\mathrm{Br}} := \tau'_{ii}, \quad \tau_e^{\mathrm{Br}} := \frac{1}{\sqrt{2}}\tau'_{ee};$$

τ_e^{Br} (Braginskii's τ_e) seems defined for equation (9).

Relaxation resistivity

In general,

$$\eta = \frac{m_{\mathrm{red}}}{e^2 n \tau_{\mathrm{slow}}}$$

where τ_{slow} is interspecies drift damping period.

For a relaxation closure that includes pair plasma ($m_i = m_e$) one could use the scalar resistivity

$$\eta = \frac{\alpha m_{\mathrm{red}}}{e^2 n \tau'},$$
$$n\tau' := \tau_0 \sqrt{m_{\mathrm{red}}}\, T'^{3/2},$$

with $\alpha \in \sqrt{2}[.5, 1]$.

Neglecting resistivity

Braginskii's closures are based on Coulomb collisions. In collisionless systems, relaxation is not really mediated by Coulomb collisions, and interspecies relaxation terms should be smaller than this.

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