A proposal for a fast, shock-capturing, adaptive high resolution, two-fluid collisionless plasma algorithm.

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1 Two-fluid nondiffusive plasma equations.

We wish to develop an efficient algorithm to model a collisionless two-fluid plasma. We want this algorithm to be **conservative** (so that shocks will move at the physically correct speed), **high resolution** (meaning second-order convergence for smooth solutions, avoiding unphysical oscillations near shocks, and avoiding excessive smearing near shocks), and **h-adaptive** (meaning that the mesh size adapts to resolve features or regimes of interest). In regions where we are not interested in resolving high-speed waves we wish to take a larger time step.

2 Strategy.

We intend to use operator splitting to decouple the modeling of the flux and source terms for the gas-dynamic and electromagnetic evolution equations.

In particular, we expect our solver to consist of the following components:

1. A finite-volume high-resolution approximate Riemann solver for the hyperbolic flux term of each fluid.
2. An implicit PDE solver to solve the advection component of Maxwell's equations which posseses the following properties:
   (a) High-order accurate when solutions are smooth.
   (b) Ability to take a large time step (in regions where we don't wish to resolve fast waves). (So this will probably need to be an implicit method, at least for multiple dimensions of space.)
   (c) Close to TVD (It's not possible for a method that satisfies the first two properties to fully achieve TVD, but we can get close; perhaps we can achieve TVB using limiters – WENO.)
3. An implicit second-order accurate ODE solver for the source terms.
   (a) An implicit second-order accurate ODE solver for the effect of the source term on the currents and electric field
   (b) A second-order update of the effect of the source terms on the energies based on the currents and electric field computed by the previous solver.
4. Some means of enforcing the divergence condition for the electric field.

3 System of equations to be solved.

The collisionless plasma equations consist of the Euler gas dynamics equations with a Lorentz source term for each of the two species (electrons and ions) coupled with Maxwell's equations for the evolution of electromagnetic field.

3.1 Definition of quantities.

To discuss the equations we wish to solve, we define the following quantities:

3.1.1 Independent variables.

t = time
$\mathbf{x}$ = position

3.1.2 Particle properties.

s = species index (i for ion, e for electron)
q_s = charge of a particle
m_s = mass of a particle

3.1.3 Gas-dynamic quantities.

n_s = particle number density
$\rho_s = m_s n_s$ = mass density
$\mathbf{v}_s$ = fluid velocity
$\mathbf{M}_s = (\rho_s \mathbf{v}_s)$ = momentum
p = pressure
$\mathcal{E}$ = gas-dynamic energy

3.1.4 Electromagnetic quantities.

$\sigma_s = q_s n_s$ = charge density
$\sigma = \sum_s \sigma_s$ = net charge density
$\mathbf{J}_s = \sigma_s \mathbf{v}_s = q_s n_s \mathbf{v}_s$ = current (charge flux)
$\mathbf{J} = \sum_s \mathbf{J}_s$ = net current
$\mathbf{B}$ = magnetic field
$\tilde{\mathbf{B}} = c\mathbf{B}$ = rescaled magnetic field
$\mathbf{E}$ = electric field

3.2 Constitutive relations.

$p = (\gamma - 1)(\mathcal{E} - \frac{1}{2}\rho v^2)$

3.3 Two-fluid plasma equations

The gas-dynamics equations are:

$$\partial_t \begin{bmatrix} \rho_s \\ \rho_s \mathbf{v}_s \\ \mathcal{E}_s \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho_s \mathbf{v}_s \\ \rho_s \mathbf{v}_s \mathbf{v}_s + p_s \underline{\underline{\delta}} \\ \mathbf{v}_s \big(\mathcal{E}_s + p_s\big) \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} \\ \mathbf{J}_s \cdot \mathbf{E} \end{bmatrix}$$

where $\quad p_s = (\gamma - 1)(\mathcal{E}_s - \dfrac{1}{2}\rho_s v_s^2)$ and $\sigma_s = \dfrac{q_s}{m_s}\rho_s$ and $\mathbf{J}_s = \dfrac{q_s}{m_s}\rho_s \mathbf{v}_s$

Written in conserved variables:

$$\partial_t \underbrace{\begin{bmatrix} \rho_s \\ \mathbf{M}_s \\ \mathcal{E}_s \end{bmatrix}}_{\text{conserved}} + \nabla \cdot \underbrace{\begin{bmatrix} \mathbf{M}_s \\ \frac{\mathbf{M}_s\mathbf{M}_s}{\rho_s} + p_s \underline{\underline{\delta}} \\ \frac{\mathbf{M}_s}{\rho_s}(\mathcal{E}_s + p_s) \end{bmatrix}}_{\text{hyperbolic flux}} = \underbrace{\begin{bmatrix} 0 \\ \frac{q_s}{m_s}(\rho_s \mathbf{E} + \mathbf{M}_s \times \mathbf{B}) \\ \frac{q_s}{m_s}\mathbf{M}_s \cdot \mathbf{E} \end{bmatrix}}_{\text{electromagnetic source}}$$

$$\text{where} \quad p_s = (\gamma - 1)\Big(\mathcal{E}_s - \frac{M_s^2}{2\rho_s}\Big)$$

Maxwell's evolution equations with constraints are:

$$\partial_t \begin{bmatrix} \tilde{\mathbf{B}} \\ \mathbf{E} \end{bmatrix} + c\nabla \times \begin{bmatrix} \mathbf{E} \\ -\tilde{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\epsilon_0}\mathbf{J} \end{bmatrix} \text{ and } \nabla \cdot \begin{bmatrix} \tilde{\mathbf{B}} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\epsilon_0}\sigma \end{bmatrix},$$

$$\text{where} \quad \mathbf{J} = \mathbf{J}_i + \mathbf{J}_e = \frac{q_i}{m_i}\mathbf{M}_i + \frac{q_e}{m_e}\mathbf{M}_e$$

$$\text{and} \quad \sigma = \sigma_i + \sigma_e = \frac{q_i}{m_i}\rho_i + \frac{q_e}{m_e}\rho_e$$

3.4 Nondimensionalization.

3.4.1 Characteristic values.

Following Shumlak and Loverich [1], we nondimensionalize by using the following characteristic variables:

- x_0 = typical length scale (???FIX)
- v_0 = typical thermal velocity of an ion
- n_0 = typical number density
- $\mathbf{B}_0$ = typical magnetic field strength
- m_i = mass of an ion
- q_0 = charge strength of an ion or electron

Write $X = X_0\widehat{X}$ for each variable X, where X_0 represents the characteristic value and $\widehat{X}$ is the nondimensionalized variable. Our fundamental choices of characteristic quantities imply the following characteristic values.

$$\begin{array}{lll}
m_s = m_0\widehat{m}_s & \text{where} & m_0 = m_i \\
 & \text{and} & \widehat{m}_s = \begin{cases} 1 & \text{if } s = i \\ \frac{m_e}{m_i} & \text{if } s = e \end{cases} \\
q_s = q_0\widehat{q}_s & \text{where} & q_0 = e \\
 & \text{and} & \widehat{q}_s = \begin{cases} 1 & \text{if } s = i \\ -1 & \text{if } s = e \end{cases} \\
t = t_0\widehat{t} & \text{where} & t_0 = \frac{x_0}{v_0} \\
\rho_s = \rho_0\widehat{\rho}_s & \text{where} & \rho_0 = m_i n_0 \\
\sigma_s = \sigma_0\widehat{\sigma}_s & \text{where} & \sigma_0 = q_0 n_0 \\
\mathbf{J}_s = \mathbf{J}_0\widehat{\mathbf{J}}_s & \text{where} & \mathbf{J}_0 = q_0 n_0 v_0 \\
p_s = p_0\widehat{p}_s & \text{where} & p_0 = \rho_0 v_0^2 = m_0 n_0 v_0^2 \\
\mathcal{E}_s = \mathcal{E}_0\widehat{\mathcal{E}}_s & \text{where} & \mathcal{E}_0 = p_0 \\
E = E_0\widehat{E} & \text{where} & E_0 = B_0 v_0
\end{array}$$

Define $\widehat{\nabla} := \nabla_{\widehat{x}}$.

3.4.2 Density and Momentum variables.

We obtain the nondimensionalized equations by making the substitution $X_s = X_0\widehat{X}_s$ for each variable X in the equations and then dividing to put all the characteristic values on the right side of the equation. When the dust settles, the net result of this is to replace every variable with its corresponding hatted variable and to put a multiplicative factor consisting of the inverse of the nondimensionalized Larmor radius on the right side of the gas-dynamics equations:

$$\partial_t \begin{bmatrix} \widehat{\rho}_s \\ \widehat{\mathbf{M}}_s \\ \widehat{\mathcal{E}}_s \end{bmatrix} + \nabla \cdot \begin{bmatrix} \widehat{\mathbf{M}}_s \\ \frac{\widehat{\mathbf{M}}_s \widehat{\mathbf{M}}_s}{\widehat{\rho}_s} + \widehat{p}_s \underline{\underline{\delta}} \\ \frac{\widehat{\mathbf{M}}_s}{\widehat{\rho}_s}(\widehat{\mathcal{E}}_s + \widehat{p}_s) \end{bmatrix} = \frac{1}{\widehat{r}_L} \begin{bmatrix} 0 \\ \frac{\widehat{q}_s}{\widehat{m}_s}(\widehat{\rho}_s \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_s \times \widehat{\mathbf{B}}) \\ \frac{\widehat{q}_s}{\widehat{m}_s}\widehat{\mathbf{M}}_s \cdot \widehat{\mathbf{E}} \end{bmatrix}$$

$$\text{where} \quad \widehat{p} = (\gamma - 1)\Big(\widehat{\mathcal{E}} - \frac{\widehat{M}^2}{2\widehat{\rho}}\Big)$$

Here $r_L := \frac{m_0 v_0}{q_0 B_0}$ is the **Larmor radius**, the radius of curvature of the circular oscillation of a charge with characteristic values of mass and charge moving at the characteristic velocity perpendicular to the characteristic magnetic field. We define $\widehat{r}_L := \frac{r_L}{x_0} = \frac{m_0 v_0}{q_0 B_0 x_0}$.

Writing out the full system of gas dynamics with both species, and using that $\widehat{q}_i = 1$, $\widehat{q}_e = -1$, $\widehat{m}_i = 1$, and $\widehat{m}_e = \frac{m_e}{m_i}$ the system of equations that we must solve is:

$$\partial_t \begin{bmatrix} \widehat{\rho}_i \\ \widehat{\rho}_e \\ \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \end{bmatrix} + \nabla \cdot \begin{bmatrix} \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \frac{\widehat{\mathbf{M}}_i \widehat{\mathbf{M}}_i}{\widehat{\rho}_i} + \widehat{p}_i \underline{\underline{\delta}} \\ \frac{\widehat{\mathbf{M}}_e \widehat{\mathbf{M}}_e}{\widehat{\rho}_e} + \widehat{p}_e \underline{\underline{\delta}} \\ \frac{\widehat{\mathbf{M}}_i}{\widehat{\rho}_i}(\widehat{\mathcal{E}}_i + \widehat{p}_i) \\ \frac{\widehat{\mathbf{M}}_e}{\widehat{\rho}_e}(\widehat{\mathcal{E}}_e + \widehat{p}_e) \end{bmatrix} = \frac{1}{\widehat{r}_L} \begin{bmatrix} 0 \\ 0 \\ \widehat{\rho}_i \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_i \times \widehat{\mathbf{B}} \\ -\frac{m_i}{m_e}(\widehat{\rho}_e \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_e \times \widehat{\mathbf{B}}) \\ \widehat{\mathbf{M}}_i \cdot \widehat{\mathbf{E}} \\ -\frac{m_i}{m_e}\widehat{\mathbf{M}}_e \cdot \widehat{\mathbf{E}} \end{bmatrix}$$

$$\widehat{p}_i = (\gamma_i - 1)\Big(\widehat{\mathcal{E}}_i - \frac{\widehat{M}_i^2}{2\widehat{\rho}_i}\Big) \quad \text{and} \quad \widehat{p}_e = (\gamma_e - 1)\Big(\widehat{\mathcal{E}}_e - \frac{\widehat{M}_e^2}{2\widehat{\rho}_e}\Big)$$

Maxwell's equations become:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} + \widehat{c}\widehat{\nabla} \times \begin{bmatrix} \widehat{\mathbf{E}} \\ -\widehat{\tilde{\mathbf{B}}} \end{bmatrix} = \frac{1}{\widehat{\lambda}_D^2 \widehat{r}_L} \begin{bmatrix} 0 \\ -\widehat{\mathbf{J}} \end{bmatrix} \text{ and } \widehat{\nabla} \cdot \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} = \frac{1}{\widehat{\lambda}_D^2 \widehat{r}_L} \begin{bmatrix} 0 \\ \widehat{\sigma} \end{bmatrix},$$

$$\text{where} \quad \widehat{\mathbf{J}} = \widehat{\mathbf{J}}_i + \widehat{\mathbf{J}}_e = \widehat{\mathbf{M}}_i - \frac{m_i}{m_e}\widehat{\mathbf{M}}_e$$

$$\text{and} \quad \widehat{\sigma} = \widehat{\sigma}_i + \widehat{\sigma}_e = \widehat{\rho}_i - \frac{m_i}{m_e}\widehat{\rho}_e$$

Here $\lambda_D := \sqrt{\frac{\epsilon_0 m_0 v_0^2}{n_0 q_0^2}}$ is the **Debye length**, which is the distance scale over which electrons screen out electric fields in plasmas (i.e. the distance scale over which significant charge separation can occur). (Recall that for an ideal gas, the mean translational kinetic energy is $\frac{1}{2}mv^2 = \frac{3}{2}kT$, where k is the Boltzmann constant and T is the temperature, so the electron and ion Debye lengths are equal for equal ion and electron temperatures.) We define $\widehat{\lambda}_D$ as the ratio of the Debye length to the ion Larmor radius: $\widehat{\lambda}_D^2 := \frac{\lambda_D^2}{r_L} = \frac{\epsilon_0 B_0^2}{n_0 m_0}$. Note that $\widehat{\lambda}_D^2 \widehat{r}_L = \frac{\epsilon_0 B_0 v_0}{n_0 q_0 x_0}$.

3.4.3 Number density and current variables

We can also rewrite equations as conservation laws for number density and current, rather than mass density and momentum. We use the relations $\rho_s = m_s n_s$ and $\mathbf{M}_s = \frac{m_s}{q_s}\mathbf{J}_s$. We multiply the conservation of mass equation by $\frac{1}{m_s}$. We multiply the conservation of momentum equation by $\frac{q_s}{m_s}$.

The gas-dynamics equations become:

$$\partial_t \underbrace{\begin{bmatrix} n_s \\ \mathbf{J}_s \\ \mathcal{E}_s \end{bmatrix}}_{\text{conserved}} + \nabla \cdot \underbrace{\begin{bmatrix} \frac{\mathbf{J}_s}{q_s} \\ \frac{\mathbf{J}_s \mathbf{J}_s}{q_s n_s} + \frac{q_s}{m_s} p_s \underline{\underline{\delta}} \\ \frac{\mathbf{J}_s}{q_s n_s}\big(\mathcal{E}_s + p_s\big) \end{bmatrix}}_{\text{hyperbolic flux}} = \underbrace{\begin{bmatrix} 0 \\ \frac{q_s}{m_s}\big(q_s n_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B}\big) \\ \mathbf{J}_s \cdot \mathbf{E} \end{bmatrix}}_{\text{electromagnetic source}}$$

$$\text{where} \quad p_s = (\gamma - 1)\Big(\mathcal{E}_s - \frac{1}{2}\frac{m_s}{q_s^2}\frac{J_s^2}{n_s}\Big)$$

Writing out the full system with both species, and using that $\widehat{q}_i = 1$, $\widehat{q}_e = -1$, $\widehat{m}_i = 1$, and $\widehat{m}_e = \frac{m_e}{m_i}$, the system of equations that we must solve is:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{n}_i \\ \widehat{n}_e \\ \widehat{\mathbf{J}}_i \\ \widehat{\mathbf{J}}_e \\ \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \end{bmatrix} + \widehat{\nabla} \cdot \begin{bmatrix} \widehat{\mathbf{J}}_i \\ -\widehat{\mathbf{J}}_e \\ \frac{\widehat{\mathbf{J}}_i \widehat{\mathbf{J}}_i}{\widehat{n}_i} + \widehat{p}_i \underline{\underline{\delta}} \\ -\frac{\widehat{\mathbf{J}}_e \widehat{\mathbf{J}}_e}{\widehat{n}_e} - \frac{m_i}{m_e}\widehat{p}_e \underline{\underline{\delta}} \\ (\widehat{p}_i + \widehat{\mathcal{E}}_i)\frac{\widehat{\mathbf{J}}_i}{\widehat{n}_i} \\ -(\widehat{p}_e + \widehat{\mathcal{E}}_e)\frac{\widehat{\mathbf{J}}_e}{\widehat{n}_e} \end{bmatrix} = \frac{1}{\widehat{r}_L} \begin{bmatrix} 0 \\ 0 \\ \widehat{n}_i \widehat{\mathbf{E}} + \widehat{\mathbf{J}}_i \times \widehat{\mathbf{B}} \\ \frac{m_i}{m_e}(\widehat{n}_e \widehat{\mathbf{E}} - \widehat{\mathbf{J}}_e \times \widehat{\mathbf{B}}) \\ \widehat{\mathbf{J}}_i \cdot \widehat{\mathbf{E}} \\ \widehat{\mathbf{J}}_e \cdot \widehat{\mathbf{E}} \end{bmatrix}$$

$$\text{where} \quad \widehat{p}_i = (\gamma_i - 1)\Big(\widehat{\mathcal{E}}_i - \frac{1}{2}\frac{\widehat{J}_i^2}{\widehat{n}_i}\Big) \quad \text{and} \quad \widehat{p}_e = (\gamma_e - 1)\Big(\widehat{\mathcal{E}}_e - \frac{1}{2}\frac{m_e}{m_i}\frac{\widehat{J}_e^2}{\widehat{n}_e}\Big);$$

Maxwell's equations are:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\mathbf{B}} \\ \widehat{\mathbf{E}} \end{bmatrix} + \widehat{c}\widehat{\nabla} \times \begin{bmatrix} \widehat{\mathbf{E}} \\ -\widehat{\mathbf{B}} \end{bmatrix} = \frac{1}{\widehat{r}_L} \begin{bmatrix} 0 \\ \frac{-1}{\widehat{\lambda}_D^2}\widehat{\mathbf{J}} \end{bmatrix} \text{ and } \widehat{\nabla} \cdot \begin{bmatrix} \widehat{\mathbf{B}} \\ \widehat{\mathbf{E}} \end{bmatrix} = \frac{1}{\widehat{r}_L} \begin{bmatrix} 0 \\ \frac{1}{\widehat{\lambda}_D^2}\widehat{\sigma} \end{bmatrix},$$

$$\text{where} \quad \widehat{\mathbf{J}} = \widehat{\mathbf{J}}_i + \widehat{\mathbf{J}}_e \quad \text{and} \quad \widehat{\sigma} = \widehat{\sigma}_i + \widehat{\sigma}_e = \widehat{n}_i - \widehat{n}_e.$$

4 Numerical solver without divergence constraint enforcement.

Our approach to solving this system of equations is to use operator splitting. We plan to develop second-order solvers for each of the following equations:

4.1 Gas-dynamics

$$\partial_{\hat{t}} \begin{bmatrix} \hat{n}_i \\ \hat{n}_e \\ \hat{\mathbf{J}}_i \\ \hat{\mathbf{J}}_e \\ \hat{\mathcal{E}}_i \\ \hat{\mathcal{E}}_e \end{bmatrix} + \hat{\nabla} \cdot \begin{bmatrix} \hat{\mathbf{J}}_i \\ -\hat{\mathbf{J}}_e \\ \frac{\hat{\mathbf{J}}_i \hat{\mathbf{J}}_i}{\hat{n}_i} + \hat{p}_i \underline{\underline{\delta}} \\ -\frac{\hat{\mathbf{J}}_e \hat{\mathbf{J}}_e}{\hat{n}_e} - \frac{m_i}{m_e} \hat{p}_e \underline{\underline{\delta}} \\ (\hat{p}_i + \hat{\mathcal{E}}_i) \frac{\hat{\mathbf{J}}_i}{\hat{n}_i} \\ -(\hat{p}_e + \hat{\mathcal{E}}_e) \frac{\hat{\mathbf{J}}_e}{\hat{n}_e} \end{bmatrix} = 0$$

We plan to use a standard explicit finite-volume shock-capturing method, such as a Roe solver with higher-order corrections.

4.2 ODE solver for source terms.

4.2.1 Interdependent source term ODE solver.

$$\partial_{\hat{t}} \begin{bmatrix} \hat{\mathbf{J}}_i \\ \hat{\mathbf{J}}_e \\ \hat{\mathbf{E}} \end{bmatrix} = \frac{1}{\hat{r}_L} \begin{bmatrix} \hat{n}_i \hat{\mathbf{E}} + \hat{\mathbf{J}}_i \times \hat{\mathbf{B}} \\ \frac{m_i}{m_e} (\hat{n}_e \hat{\mathbf{E}} - \hat{\mathbf{J}}_e \times \hat{\mathbf{B}}) \\ \frac{-1}{\hat{\lambda}_D^2} \hat{\mathbf{J}} \end{bmatrix}$$

Due to the operator splitting, we can take this as an ODE with constant coefficients. The eigenvalues of this ODE are imaginary, so we intend to use the TR-BDF2 method. It is an implicit 2-stage Runge-Kutta method based on taking a half time-step with the Trapezoidal rule and then a half step with the 2-step BDF (Backward Differentiation Formula) method:

$$U^* = U^n + \frac{k}{4}(f(U^n) + f(U^*))$$

$$3U^{n+1} - 4U^* + U^n = kf(U^{n+1}).$$

4.2.2 Dependent source term ODE solver

$$\partial_{\hat{t}} \begin{bmatrix} \hat{\mathcal{E}}_i \\ \hat{\mathcal{E}}_e \end{bmatrix} = \frac{1}{\hat{r}_L} \begin{bmatrix} \hat{\mathbf{J}}_i \cdot \hat{\mathbf{E}} \\ \hat{\mathbf{J}}_e \cdot \hat{\mathbf{E}} \end{bmatrix}$$

A second-order ODE solver for each of these energy variables is:

$$\frac{\hat{\mathcal{E}}^{n+1} - \hat{\mathcal{E}}^n}{\Delta t} = \frac{(\hat{\mathbf{J}} \cdot \hat{\mathbf{E}})^n + (\hat{\mathbf{J}} \cdot \hat{\mathbf{E}})^{n+1}}{2\hat{r}_L}$$

4.3 Maxwell's advection solver.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} + \widehat{c}\widehat{\nabla} \times \begin{bmatrix} \widehat{\mathbf{E}} \\ -\widehat{\tilde{\mathbf{B}}} \end{bmatrix} = 0$$

This is a hyperbolic system with constant coefficients.

4.3.1 Maxwell advection solver in one dimension.

Consider the one-dimensional case where all quantities depend only on $x := x^1$. Then this system becomes:

$$\partial_t \begin{bmatrix} \tilde{B}^1 \\ \tilde{B}^2 \\ \tilde{B}^3 \\ E^1 \\ E^2 \\ E^3 \end{bmatrix} + c\partial_x \underbrace{\begin{bmatrix} 0 \\ -E^3 \\ E^2 \\ 0 \\ c\tilde{B}^3 \\ -c\tilde{B}^2 \end{bmatrix}}_{\text{flux}} = 0$$

i.e.

$$\partial_t \begin{bmatrix} \tilde{B}^1 \\ \tilde{B}^2 \\ \tilde{B}^3 \\ E^1 \\ E^2 \\ E^3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{flux jacobian}} \begin{bmatrix} \tilde{B}^1 \\ \tilde{B}^2 \\ \tilde{B}^3 \\ E^1 \\ E^2 \\ E^3 \end{bmatrix}_x = 0$$

This flux jacobian has eigenvalues

$$\lambda^1 = \lambda^2 = -c$$
$$\lambda^3 = \lambda^4 = 0$$
$$\lambda^5 = \lambda^6 = c$$

with corresponding right eigenvectors

$$\begin{bmatrix} \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{r}^3 & \mathbf{r}^4 & \mathbf{r}^5 & \mathbf{r}^6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and left eigenvectors

$$\blacksquare \quad \begin{bmatrix} \mathbf{l}^1 \\ \mathbf{l}^2 \\ \mathbf{l}^3 \\ \mathbf{l}^4 \\ \mathbf{l}^5 \\ \mathbf{l}^6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

5 Divergence constraints.

Rewrite Maxwell's equations as:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} + \widehat{c}\widehat{\nabla} \times \begin{bmatrix} \widehat{\mathbf{E}} \\ -\widehat{\tilde{\mathbf{B}}} \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} 0 \\ -\widehat{\mathbf{J}} \end{bmatrix} \text{ and } \widehat{\nabla} \cdot \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} 0 \\ \widehat{\sigma} \end{bmatrix},$$

where $\epsilon := \widehat{\lambda}_D^2 \widehat{r}_L$ serves as a pseudo-permittivity.

Taking the divergence of the evolution equation gives:

$$\partial_{\hat{t}}\widehat{\nabla}\cdot\begin{bmatrix}\widehat{\widetilde{\mathbf{B}}}\\ \widehat{\mathbf{E}}\end{bmatrix}=\partial_{\hat{t}}\frac{1}{\epsilon}\begin{bmatrix}0\\ \widehat{\sigma}\end{bmatrix},$$

So if the divergence constraint is initially satisfied, it will continue to be satisfied by an exact solution, but a numerical solution is liable to drift from satisfying this constraint, resulting in unphysical solutions.

5.1 Potential formulation.

To ensure that the divergence constraint remains satisfied, we prefer to use a potential formulation of Maxwell's equations. We use the homogeneous Maxwell's equations to define potentials: Since $\nabla\cdot\mathbf{B}=0$, we can write $\mathbf{B}=\nabla\times\mathbf{A}$. Substituting this into $\nabla\times\mathbf{E}+\partial_t\mathbf{B}=0$ (Faraday's law) gives $\nabla\times(\mathbf{E}+\partial_t\mathbf{A})=0$. So we can write $\mathbf{E}+\partial_t\mathbf{A}=-\nabla\phi$. Now we substitute the potential representations

$$\begin{aligned}&\mathbf{B}=\nabla\times\mathbf{A}\\&\mathbf{E}=-\nabla\phi-\partial_t\mathbf{A}\end{aligned}$$

into the nonhomogeneous Maxwell's equations to obtain evolution equations for the potentials:

$$\begin{aligned}&-\frac{\sigma}{\epsilon}=-\nabla\cdot\mathbf{E}=\nabla\cdot(\nabla\phi+\partial_t\mathbf{A})\\&\frac{\mathbf{J}}{\epsilon}=-\partial_t\mathbf{E}+c^2\nabla\times\mathbf{B}=\partial_t(\nabla\phi+\partial_t\mathbf{A})+c^2\nabla\times\nabla\times\mathbf{A}\end{aligned}$$

i.e.

$$\begin{aligned}&-\frac{\sigma}{\epsilon}=\nabla^2\phi+\partial_t\nabla\cdot\mathbf{A}\\&\frac{\mathbf{J}}{\epsilon}=\partial_{tt}\mathbf{A}-c^2\nabla^2\mathbf{A}+\nabla(\partial_t\phi+c^2\nabla\cdot\mathbf{A})\end{aligned}$$

Now consider the imposition of the generic gauge condition $\nabla\cdot\mathbf{A}=D$. Maxwell's equations become:

$$\begin{aligned}&-\frac{\sigma}{\epsilon}=\nabla^2\phi+\partial_t D\\&\frac{\mathbf{J}}{\epsilon}=\partial_{tt}\mathbf{A}-c^2\nabla^2\mathbf{A}+\nabla(\partial_t\phi+c^2 D)\end{aligned}$$

5.1.1 Numerical drift from gauge condition.

If the numerical solution drifts from satisfying the gauge condition, solutions will become unphysical. What drift can we expect? Subtracting the time derivative of the first equation from the divergence of the second equation and using conservation of charge, $\partial_t\sigma+\nabla\cdot\mathbf{J}=0$, we get:

$$0=\partial_{tt}(\nabla\cdot\mathbf{A}-D)-c^2\nabla^2(\nabla\cdot\mathbf{A}-D)$$

The error obeys the wave equation, so we don't expect it to accumulate in one place.

5.1.2 Satisfying initial and gauge conditions.

How do we determine the initial conditions for the potential formulation? Assume that we are given initial conditions $\mathbf{B}_0, \mathbf{E}_0, \mathbf{J}_0$, and σ_0, and that D_0 and $(\partial_t D)_0$ are also given. It is necessary to ensure that the gauge condition holds initially. So we need the potential to satisfy the following conditions at time 0:

$$\nabla \cdot \mathbf{A} = D$$
$$\nabla \times \mathbf{A} = \mathbf{B}$$

This is the general problem of finding the unique vector field which decays at infinity and has given curl and divergence. To obtain the solution, seek $\mathbf{A} = \nabla \times \underline{\alpha} + \nabla \cdot \beta$. Substituting into the conditions and imposing the requirement that $\nabla \cdot \underline{\alpha} = 0$, we find that we need:

$$\nabla^2 \beta = D$$
$$\nabla^2 \underline{\alpha} = -\mathbf{B}$$

There exist unique such $\underline{\alpha}$ and β, and such $\mathbf{A}$ has the required properties.

So we can compute $\mathbf{A}_0$ from $\mathbf{B}_0$ and D_0. We also need initial conditions $(\partial_t \mathbf{A})_0$, which must satisfy:

$$\nabla \times \partial_t \mathbf{A} = -\nabla \times \mathbf{E}$$
$$\nabla \cdot \partial_t \mathbf{A} = \partial_t D$$

So we can get these from $\mathbf{E}_0$ and $(\partial_t D)_0$.

5.1.3 Lorentz gauge.

We impose the Lorentz gauge condition: $D = -\frac{1}{c^2}\partial_t \phi$. Then Maxwell's equations become:

$$\frac{\sigma}{\epsilon} = \frac{1}{c^2}\partial_{tt}\phi - \nabla^2 \phi$$
$$\frac{\mathbf{J}}{\epsilon} = \partial_{tt}\mathbf{A} - c^2 \nabla^2 \mathbf{A}$$

To ensure that the initial and gauge conditions are satisfied, we need:

$$\nabla \cdot \mathbf{A}_0 = \frac{-1}{c^2}(\partial_t \phi)_0$$
$$\nabla \times \mathbf{A}_0 = \mathbf{B}_0$$
$$\nabla \times (\partial_t \mathbf{A})_0 = -\nabla \times \mathbf{E}_0$$
$$\nabla \cdot (\partial_t \mathbf{A})_0 = (\frac{-1}{c^2}\partial_{tt}\phi)_0 = -\Big(\nabla^2 \phi + \frac{\sigma}{\epsilon}\Big)_0$$

There is still freedom in these conditions. We can require $\phi_0 = 0 = (\partial_t \phi)_0$. This gives $(\partial_t \mathbf{A})_0 = -\mathbf{E}_0$.

So we have rewritten Maxwell's equations in the form:

$$\frac{1}{\epsilon}\begin{bmatrix}c\sigma\\ \mathbf{J}\end{bmatrix} = \partial_{tt}\begin{bmatrix}\phi/c\\ \mathbf{A}\end{bmatrix} - c^2\nabla^2\begin{bmatrix}\phi/c\\ \mathbf{A}\end{bmatrix} \text{ with initial conditions:}$$

$$\begin{bmatrix}\phi/c\\ \mathbf{A}\end{bmatrix}_0 = \begin{bmatrix}0\\ -\nabla^{-2}\nabla\times\mathbf{B}_0\end{bmatrix} \text{ and } \left(\partial_t\begin{bmatrix}\phi/c\\ \mathbf{A}\end{bmatrix}\right)_0 = \begin{bmatrix}0\\ -\mathbf{E}_0\end{bmatrix}$$

We can simplify the notation a little by defining the components of the **4-vector potential** A^μ, the **current density 4-vector** J^μ, and its rescaled cousin $\tilde{J}^\mu$:

$$A^\mu := \begin{bmatrix}\phi/c\\ \mathbf{A}\end{bmatrix}, \quad J^\mu := \begin{bmatrix}c\sigma\\ \mathbf{J}\end{bmatrix}, \text{ and } \tilde{J}^\mu := \frac{1}{\epsilon}J^\mu.$$

Thus $\partial_{tt}A^\mu - c^2\nabla^2 A^\mu = \tilde{J}^\mu$.

Note that the components of A^μ evolve independently; μ is a free index.

5.1.4 The wave equation as a first-order sytem.

The potential formulation of Maxwell's equations with the Lorentz gauge is a wave equation. We can recast it as a first-order system by writing evolution equations for the derivatives of the potential. (Since the electric and magnetic field are defined in terms of the derivatives of the potential, this is precisely all we need.) Make the following definitions:

$$x^0 := t$$
$$\partial_\nu := \partial_{x^\nu}$$
$$W^\mu_\nu := c\partial_\nu A^\mu$$

We adopt the convention that latin indices assume values in $\{1,2,3\}$ (spacial indices), and Greek indices assume values in $\{0,1,2,3\}$ (space-time indices). We invoke the summation convention for repeated indices. We also make the convenient definitions:

$$V^\mu := W^\mu_0 = \partial_t A^\mu \quad \text{and}$$
$$\mathbf{W}^\mu := c\nabla A^\mu \quad \text{(i.e. } W^\mu_i := c\partial_{x^i}A^\mu\text{)}.$$

We can recover the electromagnetic field from the potential derivatives W^μ_ν by the relations:

$$\mathbf{B}^i = [\nabla\times\mathbf{A}]^i = \epsilon^{ij}{}_k\partial_j A^k$$
$$\text{so} \quad c\mathbf{B}^i = \epsilon^{ij}{}_k W^k_j$$
$$\text{and}$$
$$\mathbf{E}^i = [-\nabla\phi - \partial_t\mathbf{A}]^i = -\partial_i\phi - \partial_t\mathbf{A}^i = -c\partial_i\mathbf{A}^0 - \partial_t\mathbf{A}^i$$
$$= -W^0_i - W^i_0 = -W^0_i - V^i$$

(using that the spatial part of the metric tensor is the identity).

The wave equation and equality of mixed partials give us a first-order hyperbolic system with a source term.

$$\partial_t\begin{bmatrix}V^\mu\\ \mathbf{W}^\mu\end{bmatrix} - c\nabla\cdot\begin{bmatrix}\mathbf{W}^\mu\\ V^\mu\underline{\underline{\delta}}\end{bmatrix} = \begin{bmatrix}\tilde{J}^\mu\\ 0\end{bmatrix}$$

The initial conditions are:

$$\begin{bmatrix} V^0 \\ \mathbf{W^0} \end{bmatrix}_0 = \begin{bmatrix} \partial_t A^0 \\ c\nabla A^0 \end{bmatrix}_0 = \begin{bmatrix} \partial_t \phi / c \\ \nabla \phi \end{bmatrix}_0 = 0 \quad \text{and}$$

$$\begin{bmatrix} V^i \\ \mathbf{W^i} \end{bmatrix}_0 = \begin{bmatrix} \partial_t A^i \\ c\nabla A^i \end{bmatrix}_0 = \begin{bmatrix} -\mathbf{E}_0^i \\ -\nabla\nabla^{-2}[\nabla \times (c\mathbf{B}_0)]^i \end{bmatrix}_0$$

Writing the wave equation out in components:

$$\begin{bmatrix} W_0^\mu \\ W_1^\mu \\ W_2^\mu \\ W_3^\mu \end{bmatrix}_t - \begin{bmatrix} 0 & c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W_0^\mu \\ W_1^\mu \\ W_2^\mu \\ W_3^\mu \end{bmatrix}_{x^1} - \begin{bmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W_0^\mu \\ W_1^\mu \\ W_2^\mu \\ W_3^\mu \end{bmatrix}_{x^2} - \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W_0^\mu \\ W_1^\mu \\ W_2^\mu \\ W_3^\mu \end{bmatrix}_{x^3} = \begin{bmatrix} \tilde{J}^\mu \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

5.1.5 Full system with potential formulation.

Using the potential formulation for Maxwell's equations, the full system becomes:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{n}_i \\ \widehat{n}_e \\ \widehat{\mathbf{J}}_i \\ \widehat{\mathbf{J}}_e \\ \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \\ \widehat{V}^\mu \\ \widehat{\mathbf{W}}^\mu \end{bmatrix} + \widehat{\nabla} \cdot \begin{bmatrix} \widehat{\mathbf{J}}_i \\ -\widehat{\mathbf{J}}_e \\ \frac{\widehat{\mathbf{J}}_i \widehat{\mathbf{J}}_i}{\widehat{n}_i} + \widehat{p}_i \underline{\underline{\delta}} \\ -\frac{\widehat{\mathbf{J}}_e \widehat{\mathbf{J}}_e}{\widehat{n}_e} - \frac{m_i}{m_e} \widehat{p}_e \underline{\underline{\delta}} \\ (\widehat{p}_i + \widehat{\mathcal{E}}_i) \frac{\widehat{\mathbf{J}}_i}{\widehat{n}_i} \\ -(\widehat{p}_e + \widehat{\mathcal{E}}_e) \frac{\widehat{\mathbf{J}}_e}{\widehat{n}_e} \\ -\widehat{c}\widehat{\mathbf{W}}^\mu \\ -\widehat{c}\widehat{V}^\mu \underline{\underline{\delta}} \end{bmatrix} = \frac{1}{\widehat{r}_L} \begin{bmatrix} 0 \\ 0 \\ \widehat{n}_i \widehat{\mathbf{E}} + \widehat{\mathbf{J}}_i \times \widehat{\mathbf{B}} \\ \frac{m_i}{m_e} (\widehat{n}_e \widehat{\mathbf{E}} - \widehat{\mathbf{J}}_e \times \widehat{\mathbf{B}}) \\ \widehat{\mathbf{J}}_i \cdot \widehat{\mathbf{E}} \\ \widehat{\mathbf{J}}_e \cdot \widehat{\mathbf{E}} \\ \frac{1}{\widehat{\lambda}_D^2} \widehat{J}^\mu \\ 0 \end{bmatrix}$$

with initial conditions:

$$\begin{bmatrix} V^0 \\ \mathbf{W^0} \end{bmatrix}_0 = 0 \quad \text{and}$$

$$\begin{bmatrix} V^i \\ \mathbf{W^i} \end{bmatrix}_0 = \begin{bmatrix} -\mathbf{E}_0^i \\ -\nabla\nabla^{-2}[\nabla \times (c\mathbf{B}_0)]^i \end{bmatrix}_0$$

with constitutive relations:

$$\widehat{p}_i = (\gamma_i - 1)\Big(\widehat{\mathcal{E}}_i - \frac{1}{2}\frac{\widehat{J}_i^2}{\widehat{n}_i}\Big)$$

$$\widehat{p}_e = (\gamma_e - 1)\Big(\widehat{\mathcal{E}}_e - \frac{1}{2}\frac{m_e}{m_i}\frac{\widehat{J}_e^2}{\widehat{n}_e}\Big);$$

and with defining relations:

$$\widehat{J}^\mu = \begin{bmatrix} \widehat{c}\,\widehat{\sigma} \\ \widehat{\mathbf{J}} \end{bmatrix}$$

$$\widehat{\mathbf{J}} = \widehat{\mathbf{J}}_i + \widehat{\mathbf{J}}_e$$

$$\widehat{\sigma} = \widehat{\sigma}_i + \widehat{\sigma}_e = \widehat{n}_i - \widehat{n}_e$$

$$\mathbf{B}^i = \epsilon^{ij}{}_k W_j^k$$

$$\mathbf{E} = -\mathbf{W}^0 - \mathbf{V}$$

6 Numerical solver with divergence constraint enforcement.

To solve the full system with the potential formulation of Maxwell's equations, we can use the same gas-dynamics solver, but we need a different ODE solver for the interdependent source term, and we need a solver for the homogeneous part of the wave equation governing the components of the electromagnetic potential (rather than a solver for the homogeneous part of Maxwell's evolution equations).

6.1 ODE solver for source terms.

6.1.1 Interdependent source term ODE solver.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\mathbf{J}}_i \\ \widehat{\mathbf{J}}_e \\ \widehat{\mathbf{V}} \end{bmatrix} = \frac{1}{\widehat{r}_L} \begin{bmatrix} \widehat{n}_i \widehat{\mathbf{E}} + \widehat{\mathbf{J}}_i \times \widehat{\mathbf{B}} \\ \frac{m_i}{m_e}(\widehat{n}_e \widehat{\mathbf{E}} - \widehat{\mathbf{J}}_e \times \widehat{\mathbf{B}}) \\ \frac{1}{\widehat{\lambda}_D^2} \widehat{\mathbf{J}} \end{bmatrix} \qquad \begin{matrix} \text{where:} \\ \widehat{\mathbf{J}} = \widehat{\mathbf{J}}_i + \widehat{\mathbf{J}}_e \\ \mathbf{E} = -\mathbf{W}^0 - \mathbf{V} \end{matrix}$$

The only difference between this ODE and the ODE without the divergence constraint enforcement is that it is not homogeneous; there is a constant forcing term.

References

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