Problem 1 (20 points): On the interval $[-2, 3]$ consider the function

$$f(x) = 2x^3 - 3x^2 - 12x.$$

Determine in which intervals the function is increasing, decreasing, concave up and concave down; determine the local extrema, global extrema and the inflection points.

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

$f''(x) = 12x - 6 = 6(2x-1) = 12(x - \frac{1}{2})$

$f'(x) = 0 \iff x = 2$ or $x = -1$

$f'(x) > 0 \iff x > 2$ or $x < -1$

$f'(x) < 0 \iff -1 < x < 2$

Therefore f is increasing on $[-2, -1)$ and $(2, 3]$, and decreasing on $(-1, 2)$.

$f''(x) > 0 \iff x > \frac{1}{2}$ $\qquad$ $f''(x) < 0 \iff x < \frac{1}{2}$

Therefore f is concave up on $(\frac{1}{2}, 3]$, and concave down on $[-2, \frac{1}{2})$.

The critical points of $f \iff f'(x) = 0 \iff x = -1, 2$.
The inflection pt is $x = \frac{1}{2}$, since $f''(x)$ changes sign there.
To determine local extrema, we use 2nd derivative test:

$f''(-1) = 12(-1 - \frac{1}{2}) < 0 \Rightarrow f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1)$
$= 7$ local maximum.

$f''(2) = 12(2 - \frac{1}{2}) > 0 \Rightarrow f(2) = 2 \cdot 2^3 - 3 \cdot 2^2 - 24$
$= -20$ local minimum.

For end points, $f'(3) > 0$, so $f(3) = -9$ is a local maximum; $f'(-2) > 0$, so $f(-2) = -4$ is a local minimum

To determine global extrema, compare the values with end points:

$f(-2) = 2(-2)^3 - 3(-2)^2 + 24 = -16 - 12 + 24 = -4$

$f(3) = 2 \cdot 27 - 27 - 36 = -9$

We've computed $f(-1) = 7$, $f(2) = -20$, so $f(-1) = 7$ is the global maximum, and $f(2) = -20$ is the global minimum.

Problem 2 (5 points): Use L'Hospital's rule to evaluate

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x}$$

$$\lim_{x \to \frac{\pi}{2}} (1 - \sin x) = 1 - \sin \frac{\pi}{2} = 0$$

$$\lim_{x \to \frac{\pi}{2}} (1 + \cos 2x) = 1 + \cos \pi = 1 + (-1) = 0$$

So this is a $\frac{0}{0}$ indeterminate form.

By L'Hospital's Rule,

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x} = \lim_{x \to \frac{\pi}{2}} \frac{0 - \cos x}{0 - 2\sin 2x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{2\sin 2x}$$

Note $\lim_{x \to \frac{\pi}{2}} \cos x = 0$, $\lim_{x \to \frac{\pi}{2}} 2\sin 2x = 2\sin \pi = 0$, So it is

$\frac{0}{0}$ indeterminate form again. By L'Hospital's Rule

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{2\sin 2x} = \lim_{x \to \frac{\pi}{2}} \frac{-\sin x}{2 \cdot 2\cos 2x} = \frac{1}{4} \cdot \frac{-\sin \frac{\pi}{2}}{\cos \pi} = 0.$$

Therefore $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x} = 0$

Problem 3 (20 points): Evaluate the following indefinite integrals

(a) $\int(x^2 - \frac{2}{x^3} + \sqrt{x})dx$

$$= \frac{x^{2+1}}{2+1} - 2 \cdot \frac{1}{-3+1} x^{-3+1} + \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C$$

$$= \frac{x^3}{3} + x^{-2} + \frac{2}{3} x^{\frac{3}{2}} + C$$

(b) $\int(1 + \sec^2 x + \sin 3x)dx$

$$= x + \tan x - \frac{\cos 3x}{3} + C$$

Problem 4 (15 points): Find the area between the graph of the function $f(x) = \cos x \sin^3 x$ and the x-axis over $[0, \frac{\pi}{2}]$. Then find the average value of the function on this interval.

In $[0, \frac{\pi}{2}]$, $\cos x \geqslant 0$, $\sin x \geqslant 0$, So $f(x) = \cos x \sin^3 x \geqslant 0$.

Therefore Area $= \int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^{\frac{\pi}{2}} \cos x \sin^3 x\,dx$

$$\overset{\sin x = u}{\underset{\cos x\,dx = du}{=\!=}} \int_0^1 u^3\,du$$

$$= \frac{1}{4}u^4 \Big|_0^1$$

$$= \frac{1}{4}$$

And the average value

$$f_{av} = \frac{\int_0^{\frac{\pi}{2}} \cos x \sin^3 x\,dx}{\frac{\pi}{2} - 0} = \frac{2}{\pi} \cdot \frac{1}{4}$$

$$= \frac{1}{2\pi}$$

Problem 5 (10 points): Evaluate the following limit by writing it as a definite integral and then evaluate that integral. Make sure you show some justification for the definite integral you are using.

$$\lim_{n\to+\infty} \Sigma_{i=1}^{n} \frac{i^2}{n^3}.$$

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} = \lim_{n\to+\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \qquad (*)$$

If we interprete $\left(\frac{i}{n}\right)^2 = f(c_i)$ for $f(x) = x^2$ on $[0,1]$, where $c_i = \frac{i}{n}$ is the right end point of the i-th subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ for the partition $\{0 < \frac{1}{n} < \frac{2}{n} \cdots < \frac{n}{n} = 1\}$, and $\frac{1}{n} = \frac{1-0}{n} = \Delta x$, then $(*)$ becomes

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_0^1 f(x)\,dx = \int_0^1 x^2\,dx,$$

by the definition of definite integrals. We compute

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Therefore $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{3}$.