Review Outline for first quiz, Feb 24th

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Methods of Solution of ODEs by classification

1. Nonlinear
   (a) 1st order: $y' = f(t, y)$
      i. (§2.2) **separation of variables**
      ii. approximation methods
         A. (§1.1) direction fields
         B. (§2.7) Euler's method
         C. (§2.8) successive approximations
   (b) 2nd order: $y'' = f(t, y, y')$
   (c) nth order: $y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$
      - (§7.1) convert to a first-order system and solve
2. Linear
   (a) 1st order: $y' + p(t)y = g(t)$
      i. integrating factors
      ii. short-cuts
         A. homogeneous ($g = 0$): separate
         B. constant coefficients: $y' + by = g(t)$
            - undetermined coefficients
            - Laplace transform
   (b) 2nd order: $L[h] = y'' + p(t)y' + q(t)y = g(t)$
      i. homogeneous: g=0
         A. (§3.2) general solution is a linear combination $c_1 y_1(t) + c_2 y_2(t)$, where $y_1(t)$ and $y_2(t)$ are linearly independent solutions.
            - (§3.3) Wronskian determininant
               - definition of Wronskian
               - functions are linearly independent if Wronskian is nonzero.
               - solutions to ODE are linearly dependent if Wronskian is zero.
         B. constant coefficients: $y'' + by' + cy = 0$
            guess e^{rt}
            get $r^2 + br + c = 0$
            factor: $(r - r_1)(r - r_2) = 0$
            Three cases:
            - (§3.1) distinct real roots: r_1, r_2
            - (§3.5) repeated (real) root: $r_1 = r_2$
              Two independent solutions:
              $y_1(t) = e^{rt}$
              $y_2(t) = te^{rt}$

- (§3.4) complex conjugate pair: $r = \lambda \pm i\mu$
  Two independent real solutions:
  $y_1(t) = e^{\lambda t}\cos(\mu t)$
  $y_2(t) = e^{\lambda t}\sin(\mu t)$

C. nonconstant coefficients

- (§3.5) **reduction of order** to get a second solution: Guess $y(t) = u(t)y_1(t)$.
- (Chapter 5, not covered in this course) series solutions: plug a power series $y = \sum_{k=1}^{\infty} a_n t^n$ into the differential equation and solve the resulting difference equation.

ii. nonhomogeneous ($g \neq 0$)
Need a particular solution: L[Y]=g.

A. constant coefficients

- (§3.6) **undetermined coefficients**
  Works for nice forcing functions.
  Forcing function: $g(t) = P_n(t)e^{at}(a\cos(\beta t) + b\sin(\beta t)$ (P_n an n-th order polynomial).
  Guess: $t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)\cos(\beta t) + (B_0t^n + B_1t^{n-1} + \cdots + B_n)\sin(\beta t)]e^{at}$
  where s=number of times $\alpha + i\beta$ is a root of the characteristic equation.
- (Chapter 6, not covered on first quiz) **Laplace transform**
  Works for general forcing functions, including step functions and Dirac delta functions (spike functions).

B. nonconstant coefficients

- (§3.7) **variation of parameters**
  Given: y_1 and y_2 are solutions to $L[y] = 0$.
  Seek solution Y to $L[Y] = g$ of the form $Y = u_1(t)y_1 + u_2(t)y_2$
  subject to the requirement:
  $u_1'y_1 + u_2'y_2 = 0$
  $u_1'y_1' + u_2'y_2' = g$
  Get:
  $u_1 = -\int \frac{gy_2}{W}$
  $u_2 = \int \frac{gy_1}{W}$
  where $W = y_1y_2' - y_1'y_2$.

(c) (Chapter 4) nth order (just like 2nd order)

i. homogeneous

A. (§4.2) constant coefficients Guess e^{rt}.

- factoring characteristic polynomial over complex plane.
- Use polar form to find roots.
- Use complex conjugate roots to get real solutions.

ii. nonhomogeneous

A. (§4.3) method of undetermined coefficients

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