§10.1 p575 #3, 4, 7, 8, 11, 15

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In problems 1 through 13 either solve the given boundary value problem or else show that it has no solution.

(3) $y'' + y = 0$
$y(0) = 0,\ y(L) = 0$

General solution:

$y(x) = c_1 \cos x + c_2 \sin x$

Impose Boundary conditions:

$$\begin{cases} 0 = c_1 \\ 0 = c_1 \cos L + c_2 \sin L \end{cases}$$

So $0 = c_2 \sin L$

Case $\sin L \neq 0$: then $c_2 = 0$

Case $\sin L = 0$: then c_2 is arbitrary

$\sin L = 0 \Leftrightarrow L = k\pi,\ k \in \mathbb{Z}.$

So if L is a multiple of π, then $c_2 \sin x$ is a solution, but if L is not a multiple of π, then 0 is the only solution.

(4) $y'' + y = 0$
$y'(0) = 1$
$y(L) = 0$

General solution:

$y(x) = c_1 \cos x + c_2 \sin x$

$y'(x) = -c_1 \sin x + c_2 \cos x$

Boundary condition requirements

$$\begin{cases} 1 = c_2 \\ 0 = c_1 \cos L + c_2 \sin L \end{cases}$$

(4 continued)

So $c_1 \cos L = -\sin L$

Case $\cos L = 0$:

Then $\sin L \neq 0$, so no solution.

Case $\cos L \neq 0$:

Then $c_1 = -\dfrac{\sin L}{\cos L} = -\tan L$

So there is a unique solution.

Note $\cos L = 0$ iff $L = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}.$

So if L is an odd multiple of $\frac{\pi}{2}$, then there is no solution, but otherwise

$y(x) = -\tan L \cos x + \sin x$

(7) $y'' + 4y = \cos x$
$y(0) = 0,\ y(\pi) = 0$

General solution of homogeneous ODE, $y_h'' + 4y_h = 0$:

$y_h = c_1 \cos 2x + c_2 \sin 2x$

Particular solution of ODE

Guess $Y(x) = A\cos x + B \sin x$

So $Y'(x) = -A \sin x + B \cos x$

$Y''(x) = -A \cos x - B \sin x$

So $Y'' + 4Y = 3A \cos x + 3B \sin x$

So $B = 0,\ A = \frac{1}{3}$

General solution of ODE

$y(x) = \frac{1}{3} \cos x + c_1 \cos 2x + c_2 \sin 2x$

Boundary condition requirements

$0 = \frac{1}{3} + c_1$

$0 = -\frac{1}{3} + c_1$

So $\boxed{\text{no solution exists.}}$

(8) $y'' + 4y = \sin x$
$y(0) = 0,\ y(\pi) = 0$

General homogeneous solution of ODE:

$y_h = c_1 \cos x + c_2 \sin 2x$

Particular solution of ODE:

Guess $Y(x) = B \sin x$
$Y''(x) = -B \sin x$

So $Y'' + 4Y = 3B \sin x$.
So $B = \frac{1}{3}$

General solution of ODE:

$$y(x) = \frac{1}{3} \sin x + c_1 \cos 2x + c_2 \sin 2x$$

Boundary Condition Requirements

$$\begin{cases} 0 = c_1 \\ 0 = c_1 \end{cases}$$

So $c_1 = 0$ and $c_2 =$ anything.

So $\boxed{y(x) = \frac{1}{3} \sin x + c_2 \sin 2x}$

(11) $x^2 y'' - 2xy' + 2y = 0$
$y(1) = -1,\ y(2) = 1$

General solution:

This is a second-order homogeneous differential equation, so we need to find two independent solutions.

The coefficients are nonconstant, so guessing an exponential doesn't work.

Notice that for each n $(n = 0, 1, 2)$, the coefficient of $y^{(n)}$ is a constant multiple of x^n.

We call such an ODE "equidimensional", or an "Euler equation." You can read about these equations in Boyce & DiPrima Chapter 5 section 5.

It is possible to convert any equidimensional equation to an ODE with constant coefficients by making the transformation $x = e^{\xi}$, i.e. $\xi = \ln x$.

A more straightforward way to solve an equidimensional equation is to make the guess $\boxed{y = x^{\alpha}}$.

Then $y' = \alpha x^{\alpha - 1}$
$y'' = \alpha(\alpha - 1) x^{\alpha - 2}$.

Substituting our guess into the ODE:

$$x^2 \alpha(\alpha-1) x^{\alpha-2} - 2x\alpha x^{\alpha-1} + 2x^{\alpha} = 0$$

$$[\alpha(\alpha-1) - 2\alpha + 2]\, x^{\alpha} = 0 \quad \text{(all } x \geq 1\text{)}$$

$$\alpha(\alpha-1) - 2\alpha + 2 = 0$$

$$\alpha^2 - 3\alpha + 2 = 0$$

$$(\alpha - 2)(\alpha - 1) = 0$$

$$\alpha = 1, 2.$$

So x, x^2 are a fundamental set of solutions.

So the general solution is:

$$y(x) = c_1 x + c_2 x^2$$

Boundary condition requirements

$$\begin{cases} -1 = c_1 + c_2 \\ 1 = 2c_1 + 4c_2 \end{cases}$$

So $c_2 = \frac{3}{2}$, $c_1 = \frac{-5}{2}$

So $\boxed{y(x) = -\frac{5}{2} x + \frac{3}{2} x^2}$

In problem 15 find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

(15) $y'' + \lambda y = 0$

$y'(0) = 0, \ y(\pi) = 0$

The general solution depends on whether λ is negative, zero, or positive.

<u>If $\lambda = 0$</u>, then the general solution is a straight line, $y(x) = c_1 x + c_2$. The only such solution satisfying the boundary conditions is the trivial solution $y(x) = 0$, which by definition is not an eigenfunction.

<u>If $\lambda < 0$</u>, then the general solution is

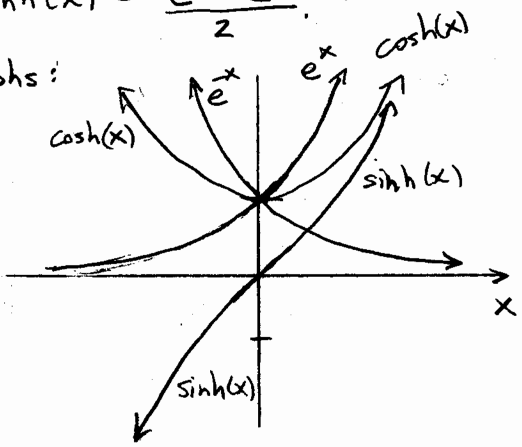
$y(x) = c_1 \cosh(\sqrt{(-\lambda)}\, t) + c_2 \sinh(\sqrt{-\lambda}\, t)$

<u>Recall hyperbolic functions:</u>

$\cosh(x) \equiv \dfrac{e^x + e^{-x}}{2}$, $\cosh'(x) = \sinh(x)$

$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$. $\sinh'(x) = \cosh(x)$

Graphs:



cosh(x) and sinh(x) are the even and odd parts of e^x.

<u>Satisfy Boundary Conditions</u>

$$\begin{cases} 0 = c_2\sqrt{-\lambda} \\ 0 = c_1 \cosh(\sqrt{-\lambda}\,\pi) + c_2 \sinh(\sqrt{-\lambda}\,\pi) \end{cases}$$

So $c_1, c_2 = 0$. So again the only solution is the trivial solution.

<u>If $\lambda > 0$</u>, then the general solution is

$y(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$y'(x) = -c_1\sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + c_2\sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$

<u>Boundary requirements</u>

$$\begin{cases} 0 = c_2\sqrt{\lambda} \quad (= y'(0)) \\ 0 = c_1 \cos(\sqrt{\lambda}\,\pi) + c_2 \sin(\sqrt{\lambda}\,\pi) \end{cases}$$

So $c_2 = 0$,

and $c_1 \cos(\sqrt{\lambda}\,\pi) = 0$

If $c_1 \neq 0$,

then $\cos(\sqrt{\lambda}\,\pi) = 0$

so $\sqrt{\lambda}\,\pi = \frac{\pi}{2}(2k+1)$, $k \in \mathbb{Z}$

So $\lambda = (k + \frac{1}{2})^2$.

Eigenvalues: $\lambda_k = (k + \frac{1}{2})^2$, k an integer (≥ 0)

corresponding eigenfunctions: (Choose $c_1 = 1$.)

$y_k(x) = \cos((k + \frac{1}{2})x)$

To make this look like the solution in the back of the book,

Let $n = k + 1$. Then:

$$\boxed{\begin{array}{l} \text{Eigenvalues: } \lambda_n = \left(\frac{2n-1}{2}\right)^2, \ n = 1, 2, \ldots \\ \text{Eigenfunctions:} \\ \quad y_n(x) = \cos\left(\frac{2n-1}{2} x\right) \end{array}}$$