Math 319, Slemrod, spring 2005.

Homework Solutions for §2.2

Covered January 18

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Problem 3. $y' + y^2 \sin x = 0$

Either $y = 0$ or:
$-\frac{y'}{y^2} = \sin x$ (separating variables)
$y^{-1} = \cos x + c$ (integrating)
$y = -1/(\cos x + c)$

Problem 4. $y' = \frac{3x^2-1}{3+2y}$

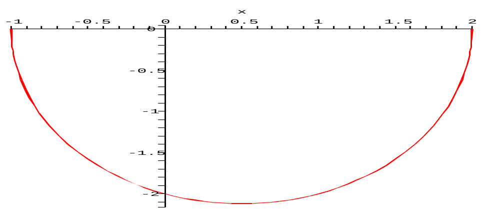
(Note that $y \neq (-3/2)$.)
$(3+2y)y' = 3x^2 - 1$ (separating)
$3y + y^2 = x^3 - x + c$

Problem 10
$y' = (1-2x)/y$
$y(1) = -2$

(a) *Find the solution of the IVP (initial value problem) in explicit form.*

Separate: $yy' = 1 - 2x$
Integrate: $y^2/2 = x - x^2 + c$
Find c using initial conditions:
$(4/2) = 1 - 1 + c$
$c = 2$
Solve for y.
$y = \pm\sqrt{2x - 2x^2 + 4}$.
Determine $\pm$ using initial conditions.
$y = -\sqrt{2x - 2x^2 + 4}$.

(b) *Plot the graph of the solution.*



(c) *Determine (at least approximately) the interval in which the solution is defined.*

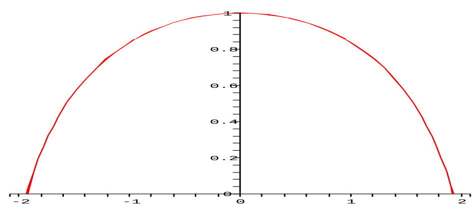
The radicand above is positive when $-1 < x < 2$.

Problem 18
$y' = \frac{e^{-x} - e^x}{3+4y}$
$y(0) = 1$

(a) *Find the solution of the IVP in explicit form.*

Separate: $(3+4y)y' = e^{-x} - e^x$ $(= -2\sinh x)$
Integrate: $3y + 2y^2 = -e^{-x} - e^x + c$ $(= -2\cosh x + c)$
(Recall that $\sinh x = \frac{e^x - e^{-x}}{2}$,
and that $\cosh x = \frac{e^x + e^{-x}}{2}$.)
Find c using initial conditions:
$5 = -2 + c$
$c = 7$
Solve for y.
$2y^2 + 3y + 2\cosh x - 7 = 0$
$y = \frac{-3 \pm \sqrt{9 - 8(2\cosh x - 7)}}{4}$
Initial condition selects positive square root.
So $y = \frac{-3}{4} + \frac{1}{4}\sqrt{65 - 16\cosh x}$
i.e. $y = \frac{-3}{4} + \frac{1}{4}\sqrt{65 - 8e^x - 8e^{-x}}$

(b) *Plot the graph of the solution.*



(c) *Determine (at least approximately) the interval in which the solution is defined.*

The radicand is positive when:
$65 - 16\cosh x > 0$
$\cosh x < 65/16$
$x < |\cosh^{-1}(65/16)| = 2.07944152$

(Recall that $\cosh^{-1} u = \ln(u + \sqrt{u^2 - 1}$, $u \geq 1$.
Indeed, let $u = \cosh x = \frac{e^x + e^{-x}}{2}$.
To solve for x, multiply by $2e^x$ to get:
$(e^x)^2 - 2ue^x + 1 = 0$

So $e^x = u \pm \sqrt{u^2 - 1}$.
Take the positive solution and solve for x:
So $x = \ln(u + \sqrt{u^2 - 1}) = \cosh^{-1} u$.

Problem 23
Solve the initial value problem

$$y\prime = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

Separate: $y\prime/y^2 = 2 + x$
Integrate: $-y^{-1} = 2x + x^2/2 + c$
Fit initial conditions: $-1 = c$
So $y = 1/(-2x - x^2/2 + 1)$.
Write $D_x = \frac{d}{dx}$.
To minimize y, set $D_x y = 0$.
Since $y = 0$ is a solution that does not satisfy our initial conditions, we know that $y(x) \neq 0$ for all x.
So $D_x y = 0$ when $D_x y^{-1} = 0$.
So set $D_x y^{-1} = 0$.
So $-2 - x = 0$.
So $x = -2$.
This maximizes y^{-1}, so minimizes y.