Section 2.8 #1-4, p.117 Andy Raich

1) $\frac{dy}{dt} = t^2 + y^2 \quad y(1) = 2.$

Let $\tilde{y} = y-2$ and $\tilde{t} = t+1$. Then $\tilde{y}(\tilde{t}) = y(t+1) - 2$ and so $\tilde{y}(0) = y(1) - 2 = 2-2 = 0.$ Since $\frac{d\tilde{y}}{dy} = 1$, $\frac{d\tilde{t}}{dt} = 1$,

$\frac{d\tilde{y}}{d\tilde{t}} = \frac{dy}{dt}$, and $\frac{d\tilde{y}}{d\tilde{t}} = (\tilde{t}-1)^2 + (\tilde{y}+2)^2$

2) $\frac{dy}{dt} = 1 - y^3$, $y(-1) = 3$

Let $\tilde{y} = y-3$ and $\tilde{t} = t-1$. Then $\tilde{y}(\tilde{t}) = y(t-1) - 3$, so $\tilde{y}(0) = y(-1) - 3 = 0.$

Also, $\frac{d\tilde{y}}{d\tilde{t}} = \frac{d\tilde{y}\ {}^{dy/d\tilde{y}}}{d\tilde{t}\ {}^{dt/d\tilde{t}}} = \frac{dy}{dt}$, so $\boxed{\frac{d\tilde{y}}{d\tilde{t}} = 1 - (\tilde{y}+3)^3}$

3) $y' = 2(y+1)$. a) $\phi_0(t) = 0$

$\phi_1(t) = \int_0^t 2(\phi_0(s)+1)\,ds = \int_0^t 2\,ds = 2t$

$\phi_2(t) = \int_0^t 2(\phi_1(s)+1)\,ds = \int_0^t 2(2s+1)\,ds = 4\frac{t^2}{2} + 2t$

$\phi_3(t) = \int_0^t 2(4\frac{s^2}{2} + 2s + 1)\,ds = \frac{8t^3}{2\cdot 3} + \frac{4t^2}{2} + 2t$

$\phi_4(t) = \int_0^t 2\left(\frac{8s^3}{2\cdot 3} + \frac{4s^2}{2} + 2s + 1\right)ds = \frac{2^4t^4}{4!} + \frac{2^3t^3}{3!} + \frac{2^2t^2}{2!} + 2t$

Claim: $\phi_n(t) = \sum_{j=1}^{n} \frac{2^j t^j}{j!}$. Base case: true for $n=1$ ($\phi_1(t) = 2t$). Assume

$\phi_{n-1}(t) = \sum_{j=1}^{n-1} \frac{2^j t^j}{j!}$. Then $\phi_n(t) = \int_0^t 2\left(\sum_{j=1}^{n-1} \frac{2^j s^j}{j!} + 1\right)ds = \sum_{j=1}^{n-1} \frac{2^{j+1}}{j!} \int_0^t s^j\,ds + 2t$

$= \sum_{j=1}^{n-1} \frac{2^{j+1}}{j!} \frac{1}{j+1} t^{j+1} + 2t = \sum_{j=1}^{n-1} \frac{2^{j+1} t^{j+1}}{(j+1)!} + 2t = \sum_{\substack{k=2 \\ k=j+1}}^{n} \frac{2^k t^k}{k!} + 2t = \sum_{k=1}^{n} \frac{2^k t^k}{k!}$

c) $\phi_n(t) = \sum_{j=1}^{n} \frac{2^j t^j}{j!}$. $\lim_{n\to\infty} \phi_n(t) = \sum_{j=1}^{\infty} \frac{(2t)^j}{j!} = \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} - 1 = \boxed{e^{2t} - 1}$

4) a) $y' = -y - 1$, $y(0) = 0$.

$\phi_0(t) = 0$. $\phi_1(t) = \int_0^t -\phi_0(s) - 1 \, ds = \int_0^t -1 \, ds = -t$

$\phi_2(t) = \int_0^t -\phi_1(s) - 1 \, ds = \int_0^t s - 1 \, ds = +\frac{1}{2}t^2 - t$

$\phi_3(t) = \int_0^t -\frac{1}{2}s^2 + s - 1 \, ds = \sum_{j=1}^{3} (-1)^j \frac{t^j}{j!}$

$\phi_4(t) = \int_0^t -\sum_{j=1}^{3} (-1)^j \frac{s^j}{j!} - 1 \, ds = \sum_{j=1}^{3} \frac{(-1)^{j+1}}{j!} \int_0^t s^j \, ds - t = \sum_{j=1}^{3} \frac{(-1)^{j+1}}{j!} \frac{t^{j+1}}{j+1} - t = \sum_{j=1}^{3} (-1)^{j+1} \frac{t^{j+1}}{(j+1)!} - t$

$= \sum_{k=2}^{4} \frac{(-1)^j t^j}{j!} - t = \sum_{k=1}^{4} \frac{(-1)^j s^j}{j!}$

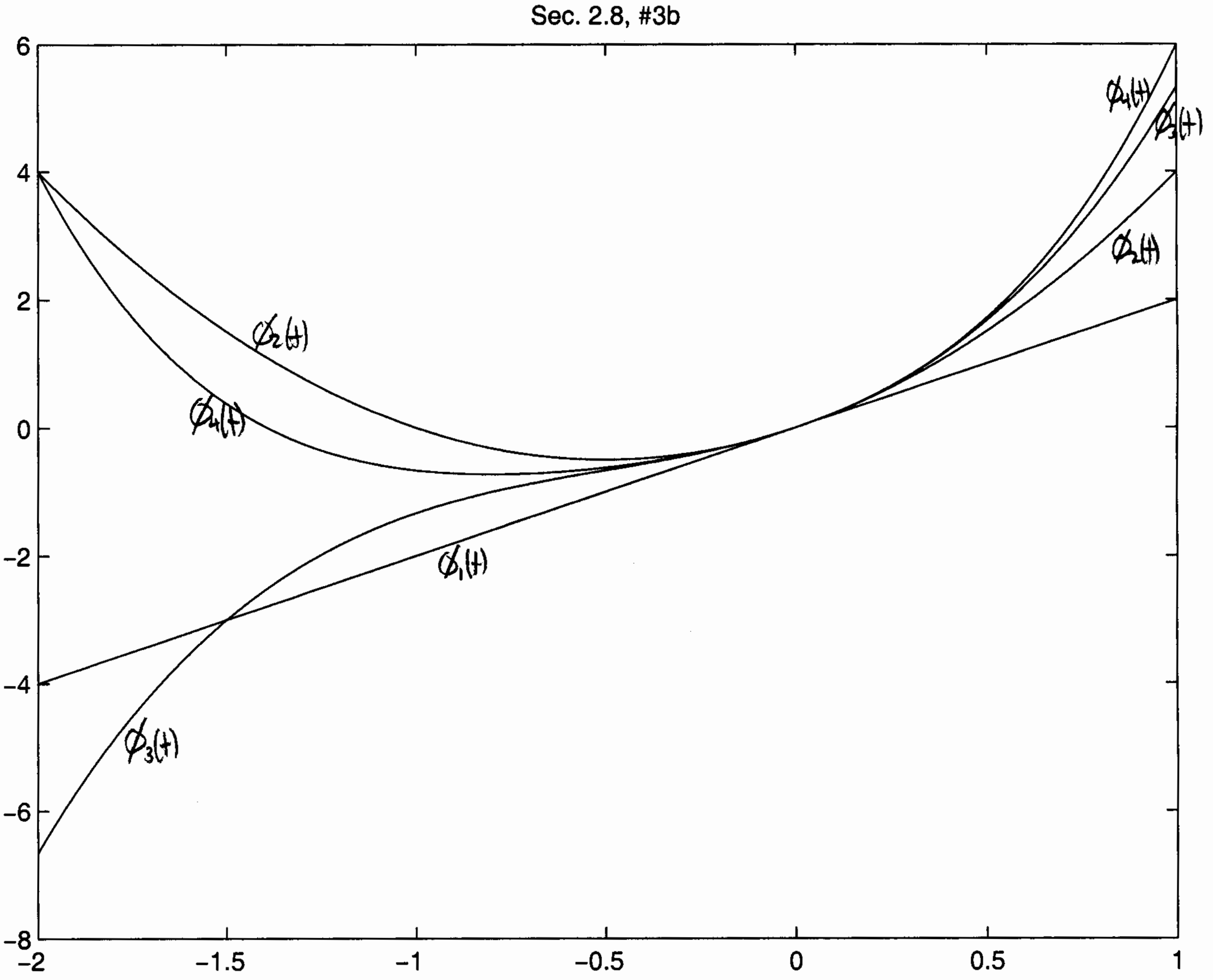
Claim/ $\phi_n(t) = \frac{(-1)^n t^n}{n!} + \frac{(-1)^{n-1} t^{n-1}}{(n-1)!} \cdots + \frac{t^2}{2} - t = \sum_{j=1}^{n} \frac{(-1)^j t^j}{j!}$

True for $\phi_1(t)$. Assume for $\phi_{n-1}(t)$. Then

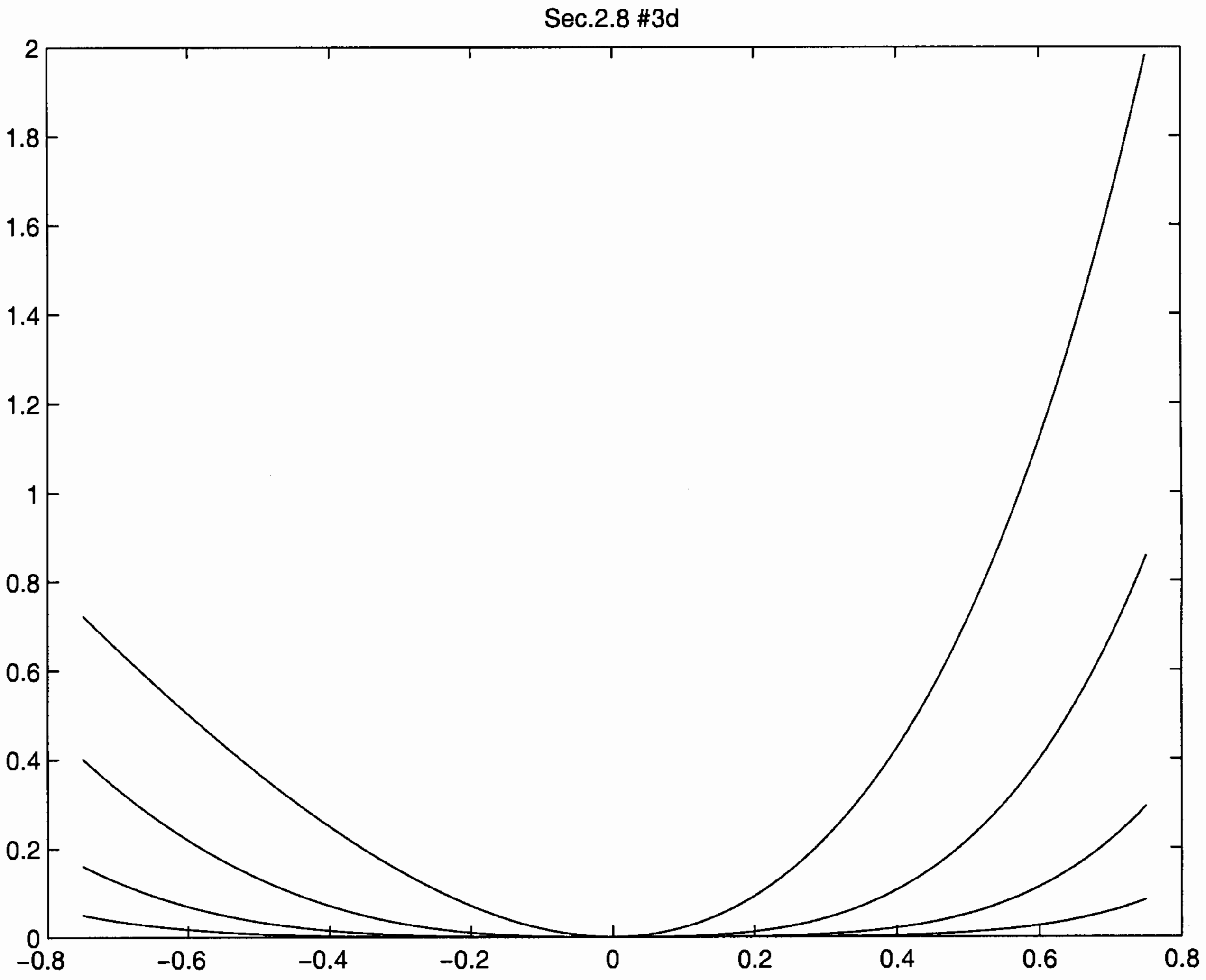
$\phi_n(t) = \int_0^t -\phi_{n-1}(s) - 1 \, ds = -\int_0^t \frac{(-1)^{n-1} s^{n-1}}{(n-1)!} + \frac{(-1)^{n-2} s^{n-2}}{(n-2)!} + \cdots - s \, ds - t$

$= \frac{(-1)^n t^n}{n!} - \frac{(-1)^{n-1} t^{n-1}}{(n-1)!} \cdots + \frac{t^2}{2} - t = \sum_{j=1}^{n} \frac{(-1)^j t^j}{j!}$

c) $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{j=1}^{\infty} \frac{(-1)^j t^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j t^j}{j!} - 1 = e^{-t} - 1$



Sec. 2.8, #3b
6
4
2
0
-2
-4
-6
-8
-2
-1.5
-1
-0.5
0
0.5
1
$\phi_4(t)$
$\phi_3(t)$
$\phi_2(t)$
$\phi_2(t)$
$\phi_4(t)$
$\phi_1(t)$
$\phi_3(t)$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1.6
1.4
1.2
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0.6
0.4
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−0.6
−0.4
−0.2
0
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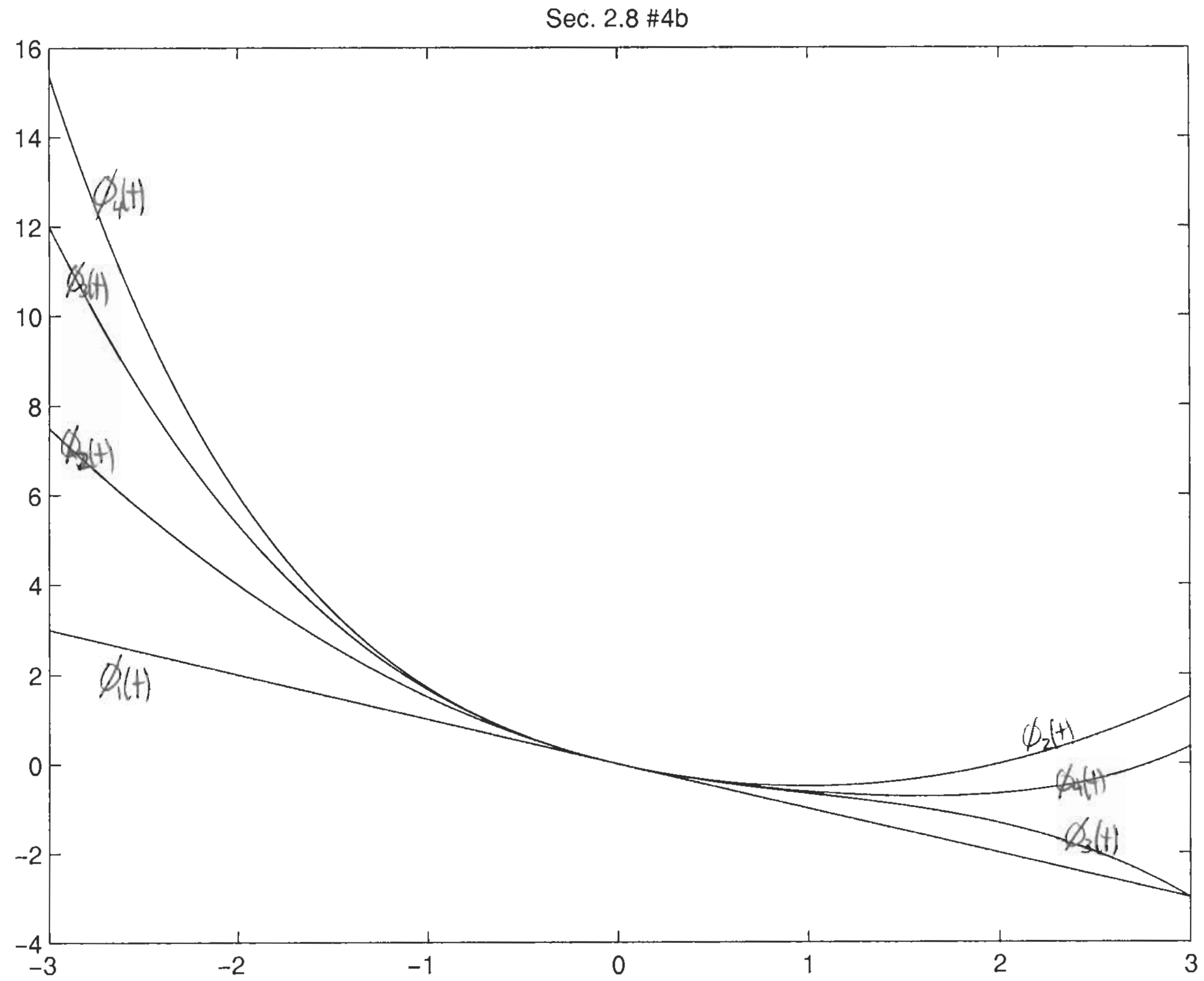
0.4
0.6
0.8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-1
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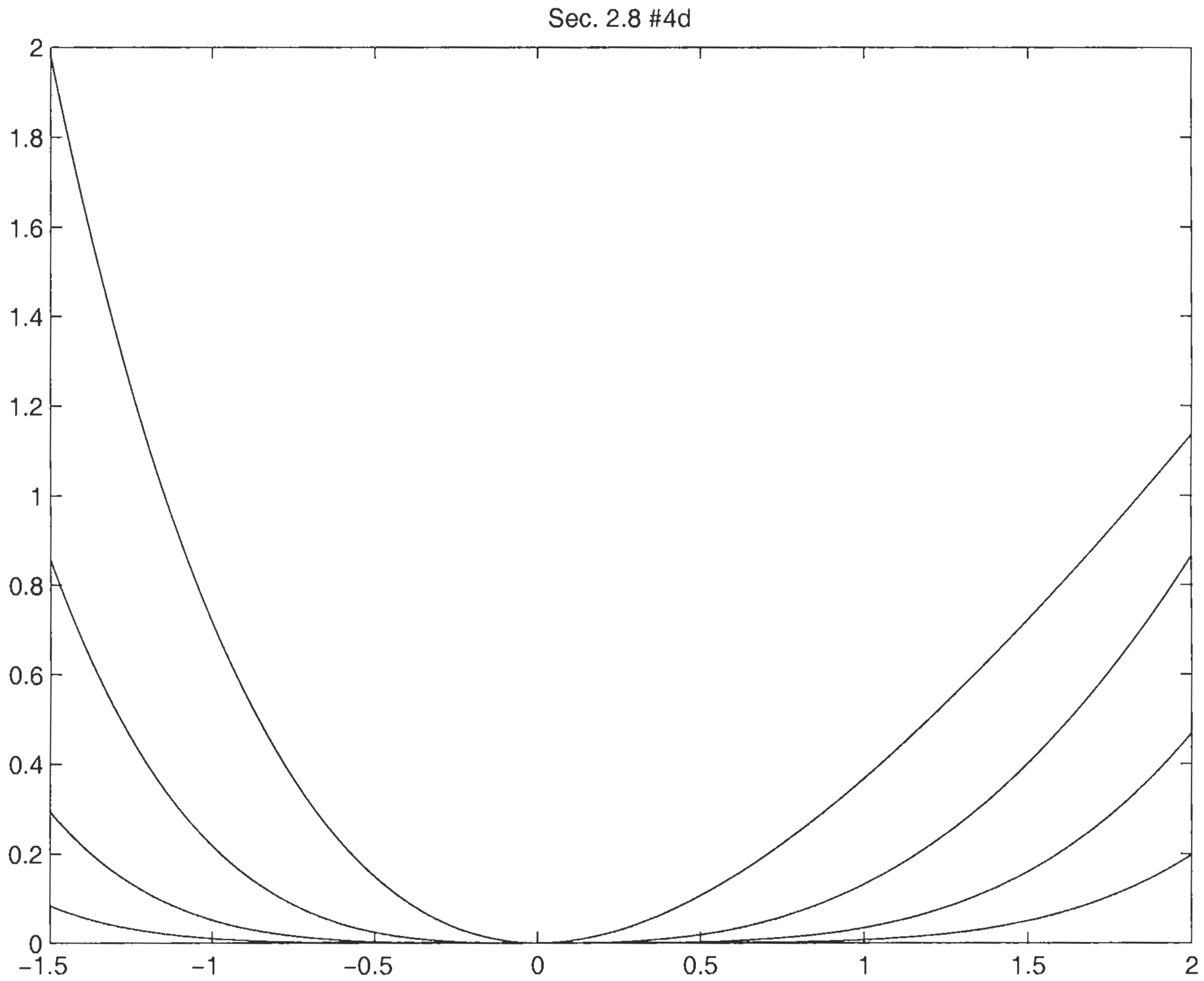
1
2
3
$\phi_4(t)$
$\phi_3(t)$
$\phi_2(t)$
$\phi_1(t)$
$\phi_2(t)$
$\phi_4(t)$
$\phi_3(t)$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1.4
1.2
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−1
−0.5
0
0.5
1
1.5
2