§3.1, covered Jan 25.
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① $y''+2y'-3y=0.$

Guess $y=e^{rt}$.
Substitute into the ODE:

$r^2e^{rt}+2re^{rt}-3e^{rt}=0$

So $r^2+2r-3=0$

So $(r+3)(r-1)=0$

So $r=-3$ or $r=1$.

So two linearly independent solutions are:

$y_1=e^{-3t}$
$y_2=e^{t}$

So the general solution is the set of linear combinations of these solutions:

$y(t)=c_1y_1(t)+c_2y_2(t)$
$=\boxed{c_1e^{-3t}+c_2e^{t}}$

② $y''+3y'+2y=0$

Guessing $y=e^{rt}$ gives

$\cancel{e^{rt}}(r^2+3r+2)=0$

$(r+1)(r+2)=0$

So $r\in\{-1,-2\}$

So $\{e^{-t},e^{-2t}\}$ are solutions.

So $\boxed{y=c_1e^{-t}+c_2e^{-2t}}$ is the general solution.

③ $6y''-y'-y=0$

Guess $y=e^{rt}$.

$6r^2-r-1=0$

$(2r-1)(3r+1)=0$

$r=\frac{1}{2}$ or $\frac{-1}{3}$

So $y=e^{\frac{1}{2}t}$ or $e^{\frac{-1}{3}t}$

General solution:

$\boxed{y=c_1e^{\frac{1}{2}t}+c_2e^{\frac{-1}{3}t}}$

④ $2y''-3y'+y=0.$

Guess e^{rt}.

$2r^2-3r+1=0$

$(2r-1)(r-1)=0$

$r\in\{\frac{1}{2},1\}$

$\boxed{y=c_1e^{\frac{1}{2}t}+c_2e^{t}}$

⑤ $y''+5y'=0$

Seek $y=e^{rt}$.

$r^2+5r=0$

$r(r+5)=0$

$r\in\{0,-5\}$

So $y\in\{e^{0t}=1,e^{-5t}\}$

Solution: $\boxed{y=c_1+c_2e^{-5t}}$,

⑥ $4y''-9y=0$

Guess e^{rt}

$4r^2-9=0$

$(2r-3)(2r+3)=0$

$r=\pm\frac{3}{2}$

$\boxed{y=c_1e^{\frac{-3}{2}t}+c_2e^{\frac{3}{2}t}}$

(§3.1)

(11) $\begin{cases} 6y'' - 5y' + y = 0 \\ y(0) = 4 \\ y'(0) = 0 \end{cases}$

(a) Characteristic equation:

$6r^2 - 5r + r = 0$

$(2r-1)(3r-1) = 0$

$r = \frac{1}{2}$ or $\frac{1}{3}$

General solution:

$y = c_1 e^{\frac{1}{2}t} + c_2 e^{\frac{1}{3}t}$

Satisfy ICs (initial conditions):

$\begin{cases} 4 = c_1 + c_2 \\ 0 = \frac{1}{2}c_1 + \frac{1}{3}c_2 \end{cases}$

$4 = c_1 + c_2$

$0 = c_1 + \frac{2}{3}c_2$

$4 = \frac{1}{3}c_2$

$\boxed{c_2 = 12}$

$\boxed{c_1 = -8}$

$$\boxed{y = -8e^{\frac{1}{2}t} + 12e^{\frac{1}{3}t}}$$

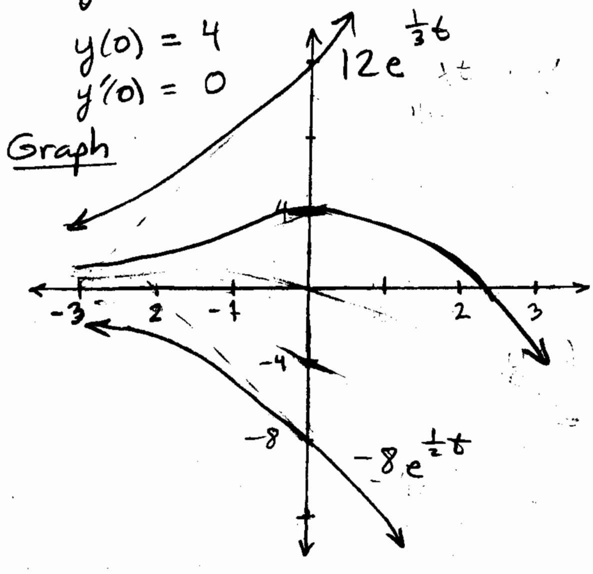
Check:

$y' = -4e^{\frac{1}{2}t} + 4e^{\frac{1}{3}t}$

$y(0) = 4$

$y'(0) = 0$

Graph



As $t \to \infty$, y looks like $-8e^{\frac{1}{2}t}$.

As $t \to -\infty$, y look like $12e^{\frac{1}{3}t}$.

(13) $\begin{cases} y'' + 5y' + 3y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$

Characteristic equation:

$r^2 + 5r + 3 = 0$

$r_\pm = \dfrac{-5 \pm \sqrt{13}}{2}$. Write $r_1 = r_+$, $r_2 = r_-$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$

Initial conditions give:

$1 = c_1 + c_2$

$0 = r_1 c_1 + r_2 c_2$

Solving:

$c_1 = \dfrac{r_2}{r_2 - r_1} = \dfrac{-5-\sqrt{13}}{-2\sqrt{13}} = \dfrac{5\sqrt{13}+13}{26}$

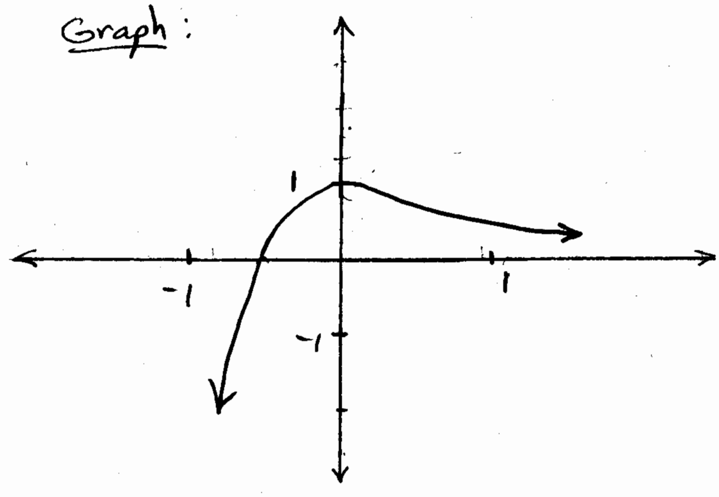
$c_2 = \dfrac{r_1}{r_1 - r_2} = 1 - c_1 = \dfrac{-5\sqrt{13}+13}{26}$

Solution:

$$\boxed{y = \left(\frac{5\sqrt{13}+13}{26}\right) e^{t\left(\frac{-5+\sqrt{13}}{2}\right)} + \left(\frac{-5\sqrt{13}+13}{26}\right) e^{t\left(\frac{-5-\sqrt{13}}{2}\right)}}$$

r_1 and r_2 are both negative, so the solution decays as $t \to \infty$.

Graph:



(§3.1)

(22) Solve the initial value problem

$$4y'' - y = 0$$
$$y(0) = 2, \quad y'(0) = \beta.$$

Characteristic equation:

$$4r^2 - 1 = 0$$
$$r_\pm = \pm \frac{1}{2}$$

$$y = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}$$
$$y' = \frac{1}{2} c_1 e^{\frac{1}{2}t} - \frac{1}{2} c_2 e^{-\frac{1}{2}t}$$

Satisfy initial conditions:

$$\begin{cases} 2 = c_1 + c_2 \\ \beta = \frac{1}{2} c_1 - \frac{1}{2} c_2 \end{cases}.$$

$$2 + 2\beta = 2c_1$$
$$\boxed{c_1 = 1 + \beta}$$
$$\boxed{c_2 = 1 - \beta}$$

Solution:

$$\boxed{y = (1+\beta) e^{\frac{1}{2}t} + (1-\beta) e^{-\frac{1}{2}t}}$$

Want: $\lim_{t \to \infty} y = 0.$

Need $1 + \beta = 0$

$$\boxed{\beta = -1}$$