§4.1 p222 #3,5,7,9,12

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In problems 3 and 5 determine intervals in which solutions are sure to exist.

③ $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$

Divide by the leading coefficient:

$$y^{(4)} + \underbrace{\left(\frac{e^t}{t(t-1)}\right)}_{\text{Call } p_2(t)} y'' + \underbrace{\left(\frac{4t}{(t-1)}\right)}_{\text{Call } p_4(t)} y = 0$$

The coefficient functions $p_2(t)$ and $p_4(t)$ are continuous except at $t=0$ and $t=1$.

So theorem 4.1.1 guarantees that on the intervals $(-\infty,0)$, $(0,1)$, and $(1,\infty)$ a unique solution exists satisfying arbitrary initial conditions.

⑤ $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$

$$y^{(4)} + \underbrace{\left(\frac{(x+1)}{(x-1)}\right)}_{p_2(x)} y'' + \underbrace{\left(\frac{\tan x}{x-1}\right)}_{p_4(x)} y = 0$$

$p_2(x)$ is continuous iff $x \neq 1$

$p_4(x)$ is continuous iff $x \neq 1$

and $\tan x \neq \pm\infty$

i.e. $\cos x \neq 0$

i.e. $x \neq \frac{\pi}{2} + k\pi$

where $k \in \mathbb{Z}$, the set of integers.

So unique solution intervals are:

$\left(\frac{\pi}{2} + (k-1)\pi, \frac{\pi}{2} + k\pi\right)$

when $k \neq 0$,

$\left(-\frac{\pi}{2}, 1\right)$ and $\left(1, \frac{\pi}{2}\right)$.

In problems 7 and 9 determine whether the given set of functions is linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

⑦ $f_1(t) = 2t-3$, $f_1'(t) = 2$, $f_1''(t) = 0$

$f_2(t) = t^2+1$, $f_2'(t) = 2t$, $f_2''(t) = 2$

$f_3(t) = 2t^2-t$, $f_3'(t) = 4t$, $f_3''(t) =$

$$W = W(f_1,f_2,f_3)(t) = \begin{vmatrix} f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{vmatrix} = \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t \\ 0 & 2 & 4 \end{vmatrix}$$

$$= (2t-3)(8t-8t) + (t^2+1)(8) + (2t^2-t)(4)$$

$$= -8t^2 - 8 + 8t^2 - 4t$$

$$= -(4t+8)$$

Since $W(t) \neq 0$ uniformly (e.g. $W(0) = -8$), The functions f_1, f_2, f_3 are linearly independent.

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9. $f_1(t) = 2t-3$
$f_2(t) = t^2+1$
$f_3(t) = 2t^2-2$
$f_4(t) = t^2+t+1$

One simple way to see that these functions are linearly dependent is that their Wronskian determinant has a row of third derivatives, and the third derivative of all these functions is uniformly zero. So the Wronskian is uniformly zero.

It is true that there are "pathological" sets of linearly independent functions who Wronskian is uniformly zero (See the comment below theorem 3.3.1 and the example of such a pair of functions given in problem 28 of §3.3).

But by theorem 4.1.2, if the set of functions is a set of solutions to a homogeneous linear differential equation whose coefficients are continuous and whose leading coefficient is 1 (or more generally, nonzero), then the Wronskian is uniformly zero if and <u>only if</u> the functions are linearly dependent.

So $f_1, \dots, f_4$ are linearly dependent.

Done.

Another way to show that these functions are linearly dependent is to find a linear combination of them with nonzero coefficients which adds to zero.

Let $l = c_1f_1(t) + c_2f_2(t) + c_3f_3(t) + c_4f_4(t)$

$$\begin{aligned} &= -3c_1 + c_2 - 2c_3 + c_4 \\ &+ (2c_1 \qquad\qquad + c_4)t \\ &+ (\qquad c_2 + 2c_3 + c_4)t^2 \end{aligned}$$

For this to be uniformly zero, need:

$$\begin{cases} -3c_1 + c_2 - 2c_3 + c_4 = 0 & (1) \\ 2c_1 \qquad\qquad + c_4 = 0 & (2) \\ \qquad c_2 + 2c_3 + c_4 = 0 & (3) \end{cases}$$

So $\boxed{c_4 = -2c_1}$

So $\begin{cases} -5c_1 + c_2 - 2c_3 = 0 \\ -2c_1 + c_2 + 2c_3 = 0 \end{cases}$

So $-7c_1 + 2c_2 = 0$

So $\boxed{c_2 = \frac{7}{2}c_1}$

Solve (3) for c_3:

$$c_3 = \frac{-c_2 - c_4}{2} = \frac{-\frac{7}{2}c_1 + 2c_1}{2}$$

$\boxed{c_3 = -\frac{3}{4}c_1}$

Choose $c_1 = 4$
So $c_2 = 14$
$c_3 = -3$
$c_4 = -8$

Verify that $c_1f_1 + \cdots + c_4f_4 = 0$.

Done.

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(12) Verify that the given functions are solutions of the differential equation, and determine their Wronskian.

$$y^{(4)} + y'' = 0$$

Solutions: $1, t, \cos t, \sin t$

<u>Check solutions:</u>

$1' = 0$

$t'' = 0$

$(\cos t)'' = -\cos t$

$(\sin t)'' = -\sin t$

<u>Wronskian</u>:

$$W(t) = \begin{vmatrix} 1 & t & \cos t & \sin t \\ 0 & 1 & -\sin t & \cos t \\ 0 & 0 & -\cos t & -\sin t \\ 0 & 0 & \sin t & -\cos t \end{vmatrix}$$

$= \cos^2 t + \sin^2 t$

$= 1.$

You can check this result using Abel's theorem:

(see §4.1 problem 20 for a proof of Abel's theorem for higher than 2^{nd}-order linear differential equations)

$$W(t) = c \exp\left(-\int p_1(t)\,dt\right)$$

where $p_1(t)$ = the coefficient of $y^{(3)}$.

But in our problem $p_1(t) = 0$.

So $W(t)$ is a constant.

But $W(0)$ is the determinant of the identity matrix, which is 1.

So $W(t) = 1$ for all t.