Section 6.2 # 5, 11, 12, 13, 14, 21, 22, 23 p.322 Andy Raich

5) Find the inverse Laplace transform.

$$\frac{2s+2}{s^2+2s+5} = \frac{2s+2}{s^2+2s+1+4} = 2\frac{(s+1)}{(s+1)^2+2^2} \qquad \begin{matrix} a=-1 \\ b=2 \end{matrix} \qquad \text{(use 9 from table)}$$

$$\mathcal{L}^{-1}\left(2\frac{(s+1)}{(s+1)^2+2^2}\right) = 2\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) = 2e^{-t}\cos(2t)$$

11) Solve: $y''-y'-6y=0$; $y(0)=1$, $y'(0)=-1$. (Assume $\phi(t)$ is a sol'n and $\mathcal{L}(\phi(t))=Y(s)$)

apply $\mathcal{L}$: $\left(s^2Y(s)-sy(0)-y'(0)\right)-\left(sY(s)-y(0)\right)-6Y(s)=0$

$$(s^2-s-6)Y(s) = s-1-1 = s-2$$

$$Y(s) = \frac{s-2}{(s-3)(s+2)} = \frac{a}{s-3}+\frac{b}{s+2}$$

$s-2=a(s+2)+b(s-3)$

try $s=-2$: $-4=-5b$, $b=\frac{4}{5}$

$s=3$: $1=5a$, $a=\frac{1}{5}$

$$Y(s) = \frac{1}{5}\frac{1}{s-3}+\frac{4}{5}\frac{1}{s+2}$$

apply $\mathcal{L}^{-1}$: $\boxed{\phi(t)=\frac{1}{5}e^{3t}+\frac{4}{5}e^{-2t}}$

12) Solve: $y''+3y'+2y=0$; $y(0)=1$, $y'(0)=0$.

Assume $\phi(t)$ is a sol'n and $\mathcal{L}(\phi(t))=Y(s)$.

Apply $\mathcal{L}$: $\left(s^2Y(s)-sy(0)-y'(0)\right)+3\left(sY(s)-y(0)\right)+2Y(s)=0$

$$(s^2+3s+2)Y(s)=s+3$$

$$Y(s)=\frac{s+3}{(s+2)(s+1)}=\frac{a}{s+2}+\frac{b}{s+1}$$

$s+3=a(s+1)+b(s+2)$

try $s=-1$: $2=b$

$s=-2$: $1=-a$, $a=-1$

$$Y(s)=\frac{-1}{s+2}+\frac{2}{s+1}$$

apply $\mathcal{L}^{-1}$: $\boxed{\phi(t)=-e^{-2t}+2e^{-t}}$

13) Solve: $y''-2y'+2y=0$; $y(0)=0$ $y'(0)=1$

Assume $\phi(t)$ is the solution and $\mathcal{L}(\phi)=Y$.

Apply $\mathcal{L}$:

$$(s^2Y(s)-sy(0)-y'(0))-2(sY(s)-y(0))+2Y(s)=0$$

$$(s^2-2s+2)Y(s)=1 \text{ , so } Y(s)=\frac{1}{(s-1)^2+1} \text{ , so } \boxed{\phi(t)=e^t\sin t}$$

$(a=1,\ b=1)$

14) Solve: $y''-4y'+4y=0$; $y(0)=1$, $y'(0)=1$

Assume $\phi(t)$ is the solution and $\mathcal{L}\phi=Y$.

Apply $\mathcal{L}$: $(s^2Y(s)-sy(0)-y'(0))-4(sY(s)-y(0))+4Y(s)=0$

$$(s^2-4s+4)Y(s)=s+1-4=s-3$$

$$Y(s)=\frac{s-3}{(s-2)^2}=\frac{s-2-1}{(s-2)^2}=\frac{1}{s-2}-\frac{1}{(s-2)^2}$$

apply $\mathcal{L}^{-1}$: $\phi(t)=e^{2t}-\mathcal{L}^{-1}\left(\frac{1!}{(s-2)^2}\right)=\boxed{e^{2t}-te^{2t}=\phi(t)}$

21) $y''-2y'+2y=\cos t$; $y(0)=1$, $y'(0)=0$

assume $\phi(t)$ is the sol'n and $\mathcal{L}\phi=Y$

Apply $\mathcal{L}$: $(s^2Y(s)-sy(0)-y'(0))-2(sY(s)-y(0))+2Y(s)=\frac{s}{s^2+1}$

$$(s^2-2s+2)Y(s)=\frac{s}{s^2+1}+s-2$$

$$Y(s)=\frac{s}{(s^2+1)(s^2-2s+2)}+\frac{s-2}{(s-1)^2+1}$$

$$\frac{s}{(s^2+1)(s^2-2s+2)}=\frac{as+b}{s^2+1}+\frac{cs+d}{s^2-2s+2}$$

$$s=(as+b)(s^2-2s+2)+(cs+d)(s^2+1)$$

s^3: $0=a+c$ $\quad$ s^2: $0=-2a+b+d$

$a=-c$

s: $1=2a-2b+c$ $\quad$ 1: $0=2b+d$

$d=-2b=\frac{4}{5}$

$1=a-2b$

$0=-2a-b$ $\quad$ $b=-2a=-\frac{2}{5}$

$a=\frac{1}{5}$ $\quad$ $c=-a=-\frac{1}{5}$

$$Y(s)=\frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{s-4}{(s-1)^2+1}+\frac{s-2}{(s-1)^2+1}$$

$$Y(s)=\frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}+\frac{4}{5}\frac{s-1}{(s-1)^2+1}-\frac{2}{5}\frac{1}{(s-1)^2+1}$$

$$\Rightarrow \boxed{\phi(t)=\frac{1}{5}\cos t-\frac{2}{5}\sin t+\frac{4}{5}e^t\cos t-\frac{2}{5}e^t\sin t}$$

22) $y'' - 2y' + 2y = e^{-t}$ $\quad y(0)=0,\ y'(0)=1$

Assume $\phi(t)$ is the solution and $\mathcal{L}\phi = Y$.

Apply $\mathcal{L}$: $\left(s^2Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{1}{s+1}$

$$(s^2-2s+2)Y(s) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{(s-1)^2+1}$$

$$\frac{1}{(s+1)(s^2-2s+2)} = \frac{a}{s+1} + \frac{bs+c}{(s^2-2s+2)}$$

$$1 = a(s^2-2s+2) + (bs+c)(s+1)$$

try $s=-1$: $1 = a(1+2+2)$, $a = \frac{1}{5}$

try $s=0$: $1 = 2(\frac{1}{5})+c$, $c = \frac{3}{5}$

s^2 terms: $0 = a+b$, $b = -\frac{1}{5}$

$$Y(s) = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\left(\frac{s-1}{(s-1)^2+1} - \frac{3}{(s-1)^2+1}\right)$$

$$\boxed{\phi(t) = \frac{1}{5}\left(e^{-t} - e^{t}\cos t + 3e^{t}\sin t\right)}$$

23) $y'' + 2y' + y = 4e^{-t}$ $\quad y(0)=2,\ y'(0)=-1$

Assume $\phi(t)$ is a sol'n and $Y(s) = \mathcal{L}\phi$.

Apply $\mathcal{L}$: $\left(s^2Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{4}{s+1}$

$$(s^2+2s+1)Y(s) = \frac{4}{s+1} + 2s - 1 + 4$$

$$(s+1)^2Y(s) = \frac{4}{s+1} + 2(s+1) + 1$$

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2}{s+1} + \frac{1}{(s+1)^2}$$

$$\phi(t) = 2t^2e^{-t} + 2e^{-t} + te^{-t}$$