§6.3 p329 #3, 5, 9, 11, 15, 20, 21, 24, 26

Alec Johnson

In problems 3 and 5 sketch the graph of the given function on the interval $t \geq 0$.

(3) $f(t-\pi)\,u_\pi(t)$ where $f(t) = t^2$

Recall that for $c \geq 0$, $u_c(t)$ denotes the Heaviside step function:

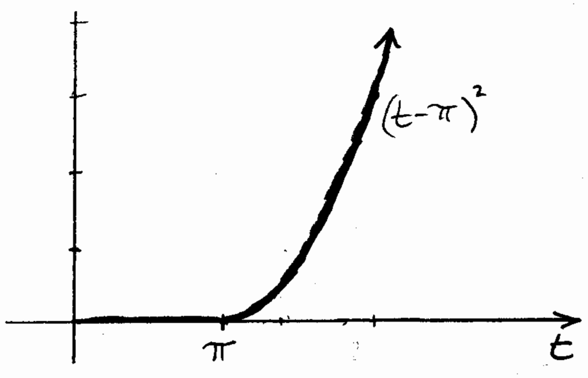
$$u_c(t) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{if } t < c \end{cases}$$

So $u_c(t) = u_0(t-c)$.

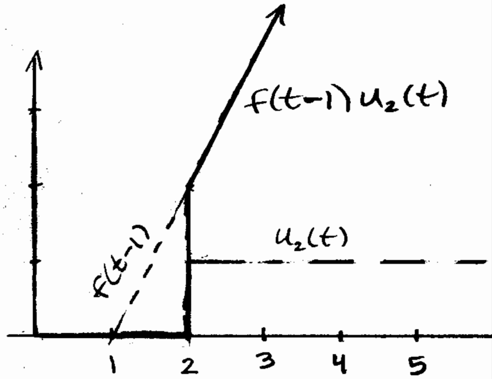
So we can write our function as:

$$f(t-\pi)\,u_0(t-\pi)$$

To graph this we shift the graph of $f(t)\,u_0(t) = \begin{cases} t^2 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$



(5) $f(t-1)\,u_2(t)$ where $f(t) = 2t$



(§6.3)

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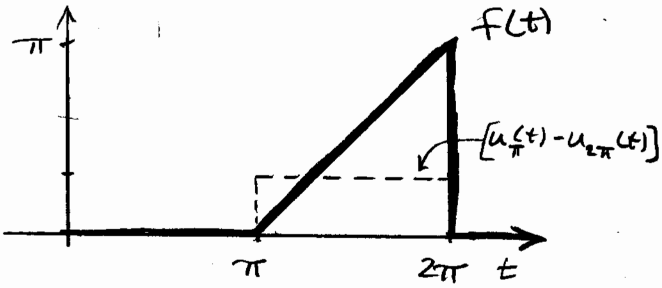
In problems 9 and 11 find the Laplace transform of the given function.

(9) $f(t) = \begin{cases} 0, & t < \pi \\ t-\pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$



Write:

$$f(t) = (t-\pi)\left(u_\pi(t) - u_{2\pi}(t)\right)$$

$$= (t-\pi)u_\pi(t) - (t-2\pi+\pi)u_{2\pi}(t)$$

$$= \left.\begin{aligned} &(t-\pi)u_\pi(t) \\ &-(t-2\pi)u_{2\pi}(t) \\ &-\pi u_{2\pi}(t) \end{aligned}\right\}$$

writing as a sum of functions whose Laplacian we know.

Recall: $\mathcal{L}\{1\} = \frac{1}{s}$,

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s)$

$\mathcal{L}\{a f(t)\} = a F(s)$

$\mathcal{L}\{f(t)+g(t)\} = F(s)+G(s)$

So:

$$F(s) = e^{-\pi s}\left(\frac{1}{s^2}\right) - e^{-2\pi s}\left(\frac{1}{s^2}\right) - \pi e^{-2\pi s}\left(\frac{1}{s}\right)$$

$$= \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2}(1+\pi s)$$

(11) $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

$$f(t) = (t-2)u_2(t) - u_2(t) - (t-3)u_3(t) - u_3(t)$$

So $F(s) = e^{-2s}\left(\frac{1}{s^2}\right) - e^{-2s}\left(\frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s^2}\right) - e^{-3s}\left(\frac{1}{s}\right)$

i.e. $F(s) = s^{-2}\left[e^{-2s}(1-s) - e^{-3s}(1+s)\right]$

(15) Find the inverse Laplace transform of

$$F(s) = \underbrace{\left(\frac{2(s-1)}{s^2-2s+2}\right)}_{\text{Call } G(s)} e^{-2s}$$

$$G(s) = \frac{2(s-1)}{(s-1)^2+1}$$

$$= 2\mathcal{L}\{e^t \cos t\}$$

So $F(s) = u_2(t) \cdot 2e^{t-2}\cos(t-2)$.

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(19) Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$

(a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right) \text{ for } s > ca.$$

$$\mathcal{L}\{f(ct)\}(s) = \int_0^\infty e^{-st} f(ct)\, dt$$

$$= \frac{1}{c} \int_0^\infty e^{-\frac{s}{c}(ct)} f(ct)\, d(ct)$$

$$= \frac{1}{c} F\left(\frac{s}{c}\right)$$

(Note that $\frac{s}{c} > a \Leftrightarrow s > ca$)

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)$$

Take the Laplace transform of both sides. This shows that we need:

$$F(ks) = \mathcal{L}\left\{\frac{1}{k} f\left(\frac{t}{k}\right)\right\}$$

$$\Leftrightarrow k F(ks) = \mathcal{L}\left\{f\left(\frac{t}{k}\right)\right\}$$

$$\Leftrightarrow \frac{1}{c} F\left(\frac{s}{c}\right) = \mathcal{L}\{f(ct)\}.$$

where $c = \frac{1}{k}$

(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a} e^{-bt/a} f\left(\frac{t}{a}\right)$$

$$\text{LHS} = \mathcal{L}^{-1}\left\{F\left(a\left(s+\frac{b}{a}\right)\right)\right\}(t)$$

$$= e^{-\frac{b}{a}t} \mathcal{L}^{-1}\{F(as)\}(t)$$

$$= e^{-\frac{b}{a}t} \frac{1}{a} f\left(\frac{t}{a}\right)$$

In problems 20 and 21 use the results of problem 19 to find the inverse Laplace transform of the given function.

(20) $F(s) = \dfrac{2^{n+1} n!}{s^{n+1}}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{n!}{\left(\frac{s}{2}\right)^{n+1}}\right\}$$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}(2t)$$

$$= 2(2t)^n$$

Easier way: $\mathcal{L}^{-1}$ is linear:

$$\mathcal{L}^{-1}\left\{2^{n+1}\frac{n!}{s^{n+1}}\right\} = 2^{n+1}\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

$$= 2^{n+1} t^n.$$

(21) $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

$$= \frac{(2s+1)}{(2s+1)^2 + 2^2}$$

Let $G(s) = \dfrac{s}{s^2+2^2}$

$$\mathcal{L}^{-1}\{G\}(t) = \cos 2t$$

$$F(s) = G(2s+1),$$

So $\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\{G(2s+1)\}(t)$

$$= e^{-\frac{1}{2}t}\left(\frac{1}{2}\cos t\right)$$

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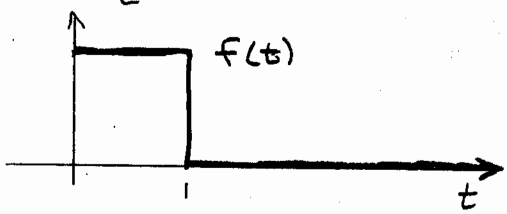
In problems 24 and 26 find the Laplace transform of the given function.

(24) $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 0 \end{cases}$



$$f(t) = u_0(t) - u_1(t)$$

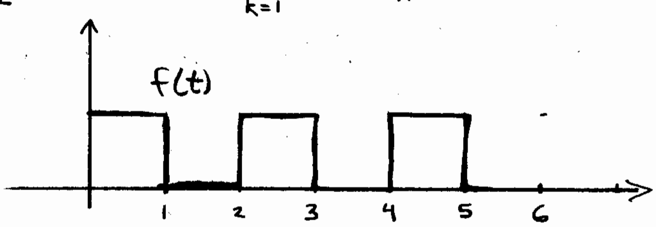
$$\mathcal{L}\{f\} = \mathcal{L}\{u_0\} - \mathcal{L}\{u_1\}$$

$$= \frac{1}{s} - \frac{e^{-s}}{s}$$

$$= \frac{1}{s}(1-e^{-s}), \quad s > 0$$

(26) $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t)$

$$= 1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t)$$



Write $w(t) = u_1(t) - u_0(t)$

So $f(t) = \sum_{\ell=1}^{n} w(t-2\ell)$

Recall from problem 26:

$$\mathcal{L}\{w\}(s) = \frac{1}{s}(1-e^{-s})$$

Recall also:

$$\mathcal{L}\{u_c(t)g(t-c)\}(s) = e^{-cs}F(s)$$

Note that

$$w(t-c) = u_c(t)\, w(t-c)$$

So $\mathcal{L}\{w(t-c)\} = e^{-cs}\left(\frac{1}{s}\right)(1-e^{-s})$

So:

$$F(s) = \sum_{\ell=0}^{n} \mathcal{L}\{w(t-2\ell)\}$$

$$= \sum_{\ell=0}^{n} e^{-2\ell s}\left(\frac{1}{s}\right)(1-e^{-s})$$

$$= \frac{1-e^{-s}}{s} \sum_{\ell=0}^{n} \underbrace{(e^{-2s})}_{\text{call } a.}{}^{\ell}$$

Recall geometric series:

Let $A = \sum_{\ell=0}^{n} a^{\ell}$

So $aA = \sum_{\ell=1}^{n+1} a^{\ell}$

So $A(1-a) = a^0 - a^{n+1}$

So $A = \dfrac{1-a^{n+1}}{1-a}$

Thus

$$F(s) = \frac{1-e^{-s}}{s}\left(\frac{1-e^{-2(n+1)s}}{1-e^{-2s}}\right), \quad s > 0$$

(see below)

(27) $f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t)$

Assume that term-by-term integration of the infinite series is permissable.

So we can take the limit of the partial sums in problem 26.

$$F(s) = \left(\frac{1-e^{-s}}{s}\right)\left(\frac{1}{1-e^{-2s}}\right)$$

$$= \frac{\cancel{(1-e^{-s})}}{\cancel{(1-e^{-s})}(1+e^{-s})}\left(\frac{1}{s}\right)$$

$$= \frac{1}{(1+e^{-s})}\left(\frac{1}{s}\right)$$

(26 cont.) Making the same cancelation in problem 26 as in problem 27 gives:

$$F(s) = \frac{1}{(1+e^{-s})}\left(\frac{1}{s}\right)\left[1 - e^{-2(n+1)s}\right]$$