§7.1 p360 # 3, 4, 10, 11

Alec Johnson

In problems 3 and 4 transform the given equation into a system of first order equations.

(3) $t^2u'' + tu' + (t^2 - 0.25)u = 0$

Let $\begin{cases} x_1 = u \\ x_2 = u' \end{cases}$

So $\begin{cases} x_1' = x_2 \\ t^2 x_2' + t x_2 + (t^2 - 0.25) x_1 = 0 \end{cases}$

Solve for the derivatives:

$$\begin{cases} x_1' = x_2 \\ x_2' = \frac{-1}{t} x_2 - \left(\frac{t^2 - 0.25}{t^2}\right) x_1 \end{cases}$$

(4) $u^{(4)} - u = 0$

Let $\begin{cases} x_1 = u \\ x_2 = u' \\ x_3 = u'' \\ x_4 = u''' \end{cases}$

So $\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = x_1 \end{cases}$

(7) Systems of first order equations can sometimes be transformed into a single equation of higher order.

Consider the system

$$x_1' = -2x_1 + x_2$$
$$x_2' = x_1 - 2x_2$$

(a) Solve the first equation for x_2

$$x_2 = x_1' + 2x_1$$

Substitute into the second equation, thereby obtaining a second order equation for x_1.

$$(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)$$

Solve this equation for x_1

$$x_1'' + 2x_1' = x_1 - 2x_1' - 4x_1$$

$$x_1'' + 4x_1' + 3x_1 = 0$$

Seek $x_1 = e^{rt}$

$$r^2 + 4r + 3 = 0$$
$$(r+3)(r+1) = 0$$

So $\boxed{x_1 = c_1 e^{-t} + c_2 e^{-3t}}$

Determine x_2 also.

$$x_2 = x_1' + 2x_1$$

$$x_1' = -c_1 e^{-t} - 3c_2 e^{-3t}$$

So $x_2 = -c_1 e^{-t} - 3c_2 e^{-3t} + 2c_1 e^{-t} + 2c_2 e^{-3t}$

$$\boxed{x_2 = c_1 e^{-t} - c_2 e^{-3t}}$$

(b) Find the solution of the given system that also satisfies the initial conditions $x_1(0) = 2$, $x_2(0) = 3$

So $\begin{cases} 2 = c_1 + c_2 \\ 3 = c_1 - c_2 \end{cases}$

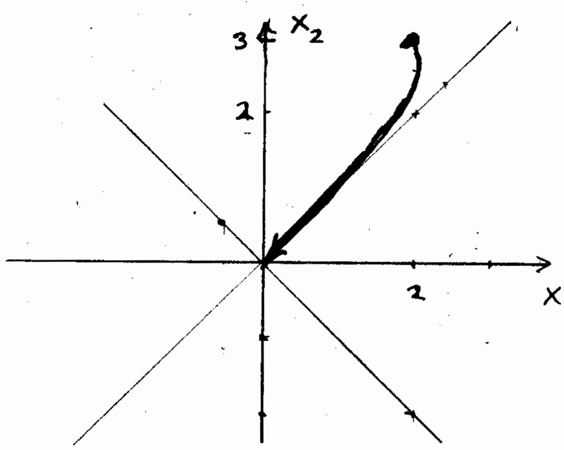
So $c_1 = \frac{5}{2}$, $c_2 = \frac{-1}{2}$

So $\begin{cases} x_1 = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t} \\ x_2 = \frac{5}{2} e^{-t} + \frac{1}{2} e^{-3t} \end{cases}$

(§7.1)

(Problem 7 cont.)

(c) Sketch the curve, for $t \ge 0$, given parametrically by the expression for x_1 and x_2 obtained in part (b).



In each of problems 10 and 11 proceed and as in problem 7 to transform the given system into a single equation of second order. Then find x_1 and x_2 that also satisfy the given initial conditions. Finally, sketch the graph of the solution in the x_1x_2-plane for $t \ge 0$.

(10) $x_1' = x_1 - 2x_2 \qquad x_1(0) = -1$

$x_2' = 3x_1 - 4x_2 \qquad x_2(0) = 2$

$$x_2 = \frac{x_1 - x_1'}{2}$$

$$\left(\frac{x_1 - x_1'}{2}\right)' = 3x_1 - 4\left(\frac{x_1 - x_1'}{2}\right)$$

$$x_1' - x_1'' = 6x_1 - 4x_1 + 4x_1'$$

$$x_1'' + 3x_1' + 2x_1 = 0$$

Guess $x_1(t) = e^{rt}$.

$r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

So $\boxed{x_1 = c_1 e^{-t} + c_2 e^{-2t}}$

$x_1' = -c_1 e^{-t} - 2c_2 e^{-2t}$

So $x_2 = \dfrac{x_1 - x_1'}{2}$ gives

$$\boxed{x_2 = c_1 e^{-t} + \tfrac{3}{2} c_2 e^{-2t}}$$

Satisfy initial conditions:

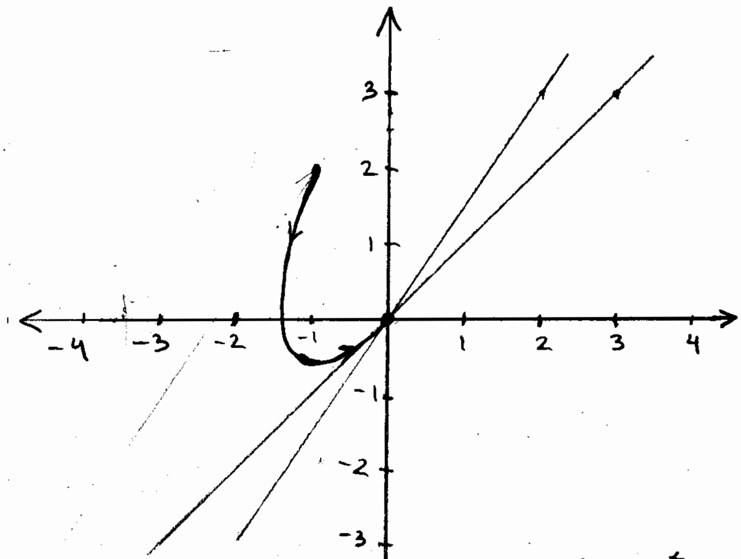
$$\begin{cases} -1 = c_1 + c_2 \\ 2 = c_1 + \frac{3}{2} c_2 \end{cases}$$

So $c_2 = 6$, $c_1 = -7$

So:

$$\boxed{\begin{aligned} x_1 &= -7e^{-t} + 6e^{-2t} \\ x_2 &= -7e^{-t} + 9e^{-2t} \end{aligned}}$$

i.e. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -7e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$



I sketched by noting that the e^{-t} component dies away less quickly and that the initial direction is given by:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}\Bigg|_{t=0} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$$

(11)

$$x_1' = 2x_2 \qquad x_1(0) = 3$$
$$x_2' = -2x_1 \qquad x_2(0) = 4$$

$$x_2 = \tfrac{1}{2}x_1'$$

$$\tfrac{1}{2}x_1'' = -2x_1$$

$$x_1'' = -4x_1$$

$$\boxed{x_1 = c_1\cos(2t) + c_2\sin(2t)}$$

$$x_1' = -2c_1\sin(2t) + 2c_2\cos(2t)$$

$$x_2 = \tfrac{1}{2}x_1'$$

$$\boxed{x_2 = -c_1\sin(2t) + c_2\cos(2t)}$$

Fit initial conditions:

$$3 = c_1$$
$$4 = c_2$$

So:

$$\boxed{\begin{aligned} x_1 &= 3\cos(2t) + 4\sin(2t) \\ x_2 &= -3\sin(2t) + 4\cos(2t) \end{aligned}}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + 4\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$$

This is a sum of two orthogonal vectors that are rotating clockwise.

Initial direction of motion:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 2\cdot 4 \\ -2\cdot 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$$