Section 7.4 p.389 #1,3,5,6 Andy Raich

1) Prove: If $\vec{x}_1, \dots, \vec{x}_k$ solve $\vec{x}' = \vec{P}(t)\vec{x}$, so does $\vec{x} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$ for any k.

Proof/ Let $\vec{x} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$ where $c_1, \dots, c_k \in \mathbb{R}$ and $\vec{x}_1, \dots, \vec{x}_k$ satisfy $\vec{x}_j' = P(t)\vec{x}_j$ for $j = 1, \dots, k$. Then:

$$\vec{x}' = \frac{d}{dt}(c_1\vec{x}_1 + \cdots + c_k\vec{x}_k) = c_1\vec{x}_1' + \cdots + c_k\vec{x}_k'$$
$$= c_1 P(t)\vec{x}_1 + \cdots + c_k P(t)\vec{x}_k = P(t)(c_1\vec{x}_1 + \cdots + c_k\vec{x}_k) = P(t)\vec{x}. \square$$

3) Show that the Wronskian of two fundamental sets of solutions of $\vec{x}' = \vec{P}(t)\vec{x}$

Proof/ By (14), if $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ and $\vec{y}^{(1)}, \dots, \vec{y}^{(n)}$ are both fundamental sets of solutions to $\vec{x}' = \vec{P}(t)\vec{x}$, then if $W_1 = W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}]$ and $W_2 = W[\vec{y}^{(1)}, \dots, \vec{y}^{(n)}]$,

$$\frac{dW_1}{dt} = (p_{11} + \cdots + p_{nn})W_1, \quad \frac{dW_2}{dt} = (p_{11} + \cdots + p_{nn})W_2.$$

So: with $j = 1$ or 2, $\frac{1}{W_j}\frac{dW_j}{dt} - (p_{11} + \cdots + p_{nn}) = 0$, so

$$\int \frac{1}{W_j}\frac{dW_j}{dt}\,dt - \int (p_{11} + \cdots + p_{nn})\,dt = c_j$$

$$\ln|W_j| = \int p_{11} + \cdots + p_{nn}\,dt + c_j, \text{ so}$$

$$W_j(t) = \tilde{c}_j e^{\int p_{11} + \cdots + p_{nn}\,dt}$$

Thus, $\frac{W_2}{W_1} = \frac{\tilde{c}_2}{\tilde{c}_1} = \tilde{\tilde{c}}$, s W_2 and W_1 are multiples

5) Show that the general solution of $\vec{x}' = \vec{P}(t)\vec{x} + \vec{g}(t)$ is the sum of any particular solution $\vec{x}^{(p)}$ of this equation and the general solution $\vec{x}^{(c)}$ of the corresponding homogeneous solution.

Proof| We must show that if $\vec{x}$ solves $\vec{x}' = \vec{P}(t)\vec{x} + \vec{g}(t)$, then $\vec{x}'$ can be written as $\vec{x} = \vec{x}^{(p)} + \vec{x}^{(h)}$ where $\vec{x}^{(p)}$ is the particular solution given in the problem and $\vec{x}^{(h)}$ is a solution to the corresponding homogeneous solution. Let $\vec{x}^{(h)} = \vec{x} - \vec{x}^{(p)}$. Then

$(\vec{x} - \vec{x}^{(p)})' = \vec{x}' - (\vec{x}^{(p)})' = \vec{g}(t) - \vec{g}(t) = 0,$

so $\vec{x} - \vec{x}^{(p)}$ is a solution to the homogeneous solution. Thus,

$$\vec{x} = \vec{x}^{(p)} + (\vec{x} - \vec{x}^{(p)})$$

is the desired decomposition.

6) Let $\vec{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\vec{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$

(a) Compute $W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$

(b) $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are L.I. at every $t \neq 0$ and on every interval.

(c) $\frac{dW}{dt} = (P_{11} + P_{22})W$, so $2t = (P_{11} + P_{22})t^2$, so $P_{11} + P_{22} = \frac{2}{t}$, so there must be a discontinuous coefficient at $t=0$.

(d) $X'(t) = P(t)X$ $\quad X(t) = (x_1(t)\ x_2(t)) = \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}$, so $X'(t) = \begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix}$

and $P(t) = X'(t)X^{-1} = \begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{2}{t} & -1 \\ -\frac{1}{t^2} & \frac{1}{t} \end{pmatrix} = \begin{pmatrix} \frac{2}{t} - \frac{2}{t} & -1+2 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix}}$