§7.5 p398 #3, 5, 7, 16, 17, 24, 25

Alec Johnson

(3) $\vec{x}' = A\vec{x}$

$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

Seek solution of form

$\vec{x} = \vec{\xi} e^{rt}$

$\vec{x}' = r\vec{\xi} e^{rt}$

So $r\vec{\xi} e^{rt} = A\vec{\xi} e^{rt}$

$A\xi = r\xi$

So r is an eigenvalue and $\vec{\xi}$ is an eigenvector.

Let I = identity matrix.

So $A\xi = rI\xi$

i.e. $(A - Ir)\xi = 0$

There is a nonzero ξ satisfying this equation only if $(A-Ir)$ is a noninvertible matrix, i.e. the determinant is zero:

$$\det(A - Ir) = 0$$

i.e. $\det\left[\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$

i.e. $\det \underbrace{\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix}}_{(A-Ir)} = 0$

i.e. $(3-r)(-2-r) - 2(-2) = 0$

$(r+2)(r-3) + 4 = 0$

i.e. $r^2 - r - 2 = 0$

i.e. $(r-2)(r+1) = 0$

Need eigenvectors.

Need $\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix}\vec{\xi} = 0$

$r_1 = 2$: $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\vec{\xi}_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

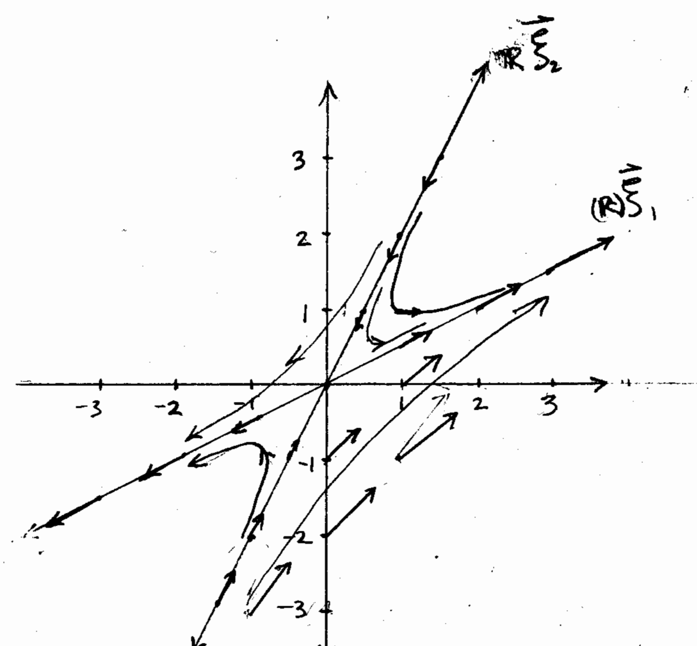
$r_2 = -1$: $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{\xi}_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So the general solution is:

$\vec{x}(t) = c_1\vec{\xi}_1 e^{r_1 t} + c_2\vec{\xi}_2 e^{r_2 t}$

$= c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$

Direction field and trajectories



Example of calculating a direction:

$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(§7.5)

3 $\vec{x}' = A\vec{x}$

$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

Eigenvalues:

Set $\underbrace{\det(A - Ir)}$ $= 0$

Call $P(r)$, the characteristic polynomial.

There is a shortcut to find the characteristic polynomial of a 2×2 matrix:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$P(r) = \det(A - Ir)$$
$$= \det \begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix}$$
$$= (a-r)(d-r) - bc$$
$$= r^2 - (a+d)r + (ad - bc)$$

So $\boxed{P(r) = r^2 - tr(A)\, r + \det(A)}$,

where $tr(A)$ = trace of A

= sum of diagonal entries

$= a + d$.

So the characteristic equation for problem number 3 is:

$r^2 - 1 = 0$

$\boxed{r = \pm 1}$

Eigenvectors:

$r = 1$: $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\xi_1} = 0$

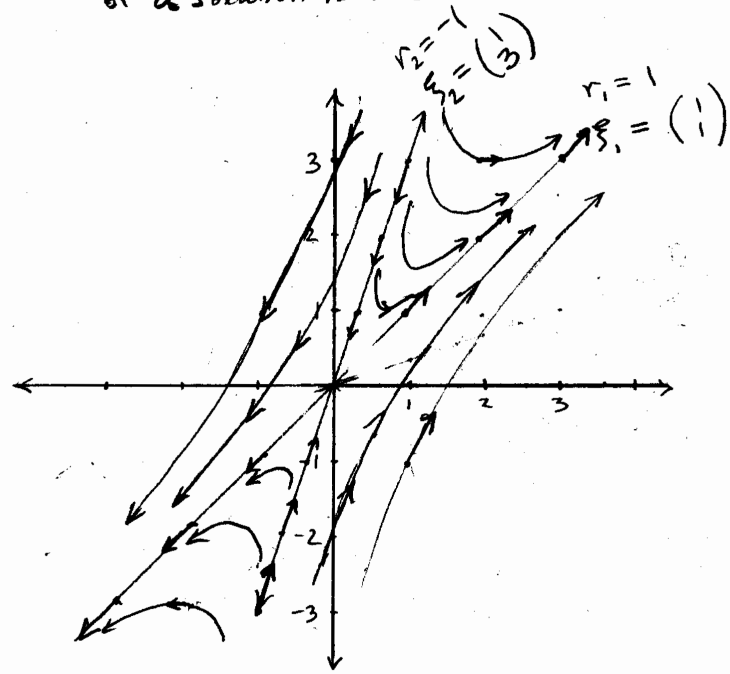
$r = -1$: $\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\xi_2} = 0$

General solution:

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

Direction field and trajectories:

Note that any scalar multiple of a solution is a solution, and any reflection through the origin of a solution is a solution.



$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

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⑤ $x' = Ax$

$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

Characteristic Polynomial:

$P(r) = r^2 - tr(A)r + det(A) = 0$

$r^2 + 4r + 3 = 0$

$(r+1)(r+3) = 0$

Eigenvectors:

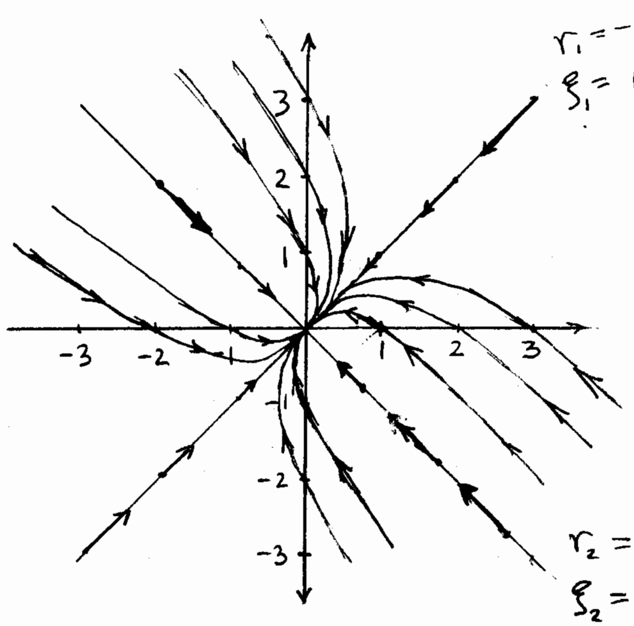
$r_1 = -1: \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{\xi_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$r_2 = -3: \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\xi_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

General Solution:

$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$

Direction field & trajectories



$r_1 = -1$
$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_2 = -3$
$\xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

⑦ $x' = Ax$

$A = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

Characteristic Polynomial:

$r^2 + 2r$
$= r(r+2)$

Eigenvectors

$r_1 = 0: \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

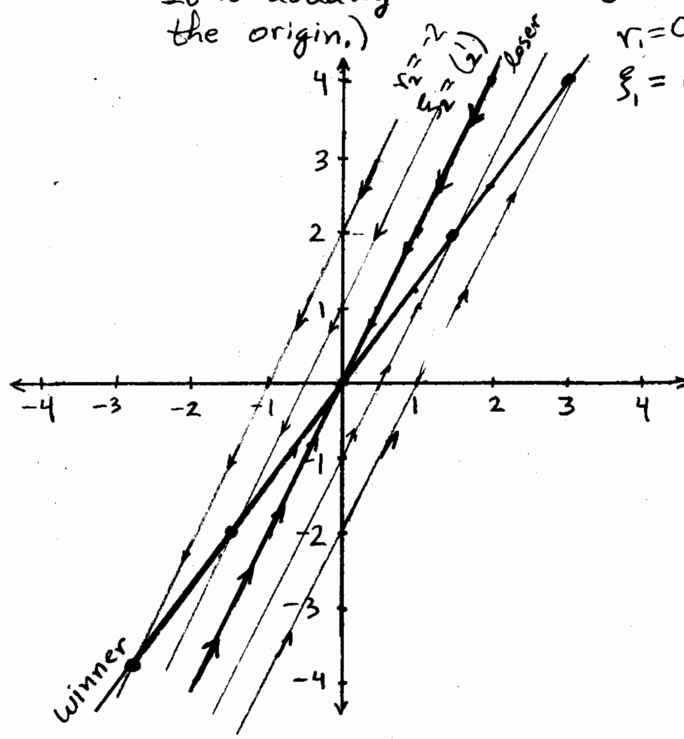
$r_2 = (-2): \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

General Solution:

$x(t) = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$

Direction field & trajectories

Points on the eigenspace corresponding to the zero eigenvector are fixed points. (The eigenspace for a given eigenvalue is the set of all eigenvectors having that eigenvalue. It is usually a line through the origin.)



$r_1 = 0$
$\xi_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

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16 $\begin{cases} x' = Ax, & A = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \\ x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{cases}$

Characteristic Equation

$r^2 - 2r - 3 = 0$

$(r+1)(r-3) = 0$

Eigenvectors

$r_1 = -1$: $\begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$r_2 = 3$: $\begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

General solution

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$$

Fit initial conditions

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

i.e. $\begin{cases} 1 = c_1 + c_2 \\ 3 = c_1 + 5c_2 \end{cases}$

So $c_2 = \frac{1}{2}$

$c_1 = \frac{1}{2}$

Solution

$$x(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$$

Behavior as $t \to \infty$.

$\lim_{t \to \infty} e^{-t} = 0$,

So the solution asymptotically approaches the "eigensolution" $\frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$.

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17 $x' = Ax$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}, \quad x(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Eigenvalues

$$\det(A - Ir) = \det \begin{pmatrix} (1-r) & 1 & 2 \\ 0 & (2-r) & 2 \\ -1 & 1 & (3-r) \end{pmatrix}$$

$$= (1-r)[(2-r)(3-r) - 2]$$
$$-1\ [0+2]$$
$$+2\ [0 + (2-r)]$$
$$= (1-r)[4 - 5r + r^2] + 2 - 2r$$
$$= 6 - 11r + 6r^2 - r^3$$

Want $r^3 - 6r^2 + 11r - 6 = 0$.

Seek rational roots. Try $\pm\{1, 2, 3, 6\}$.

$r = 1$ works. Divide.

$$\begin{array}{r} r^2 + 5r + 6 \\ r-1 \overline{)\, r^3 - 6r^2 + 11r - 6} \\ \underline{r^3 - r^2 \qquad\qquad} \\ -5r^2 + 11r \quad \\ \underline{-5r^2 + 5r \quad} \\ 6r - 6 \end{array}$$

$r^2 - 5r + 6 = (r-2)(r-3)$.

So $\det(A - Ir) = (r-1)(r-2)(r-3)$

Eigenvectors

$r_1 = 1$: $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So $b = -2c$,

$a = b + 2c = 0$

c : free to choose.

So $\xi_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} c$, set $c = 1$.

$$\boxed{\xi_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}$$

(Eigenvectors)

$r_2 = 2$: $\begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Add first row to last.

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $c = 0$, and $a = b$.

So $\boxed{\xi_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$ works.

$r_3 = 3$: $\begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Multiply last row by two, subtract 1st row, and add 2nd row.

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$b = 2c$

$a = \dfrac{b + 2c}{2} = b$

So $\xi_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

Check: $A\xi_3 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix} = 3\xi_3$ ✓

General solution

$$x = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{3t}$$

Satisfy initial conditions

Need $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

Solve: $c_1 = 1$, $c_2 = 2$, $c_3 = 0$.

$$\text{So } x = \underbrace{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{t}}_{\text{("dominates" as } t \to -\infty)} + 2 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}}_{\text{dominates as } t \to \infty.}$$

(24) Let $x' = Ax$
where the eigenvectors and eigenvalues of A are:

$$r_1 = -1, \quad \xi^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$r_2 = -2, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(a) Sketch phase portrait
(b) Sketch trajectory through $(2,3)$

Notice that it is possible to recover the matrix A:

Let $T = [\xi^{(1)}\ \xi^{(2)}] = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$
the matrix whose columns are the eigenvectors, and

let $R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$,
the diagonal matrix whose diagonal entries are the corresponding eigenvalues.

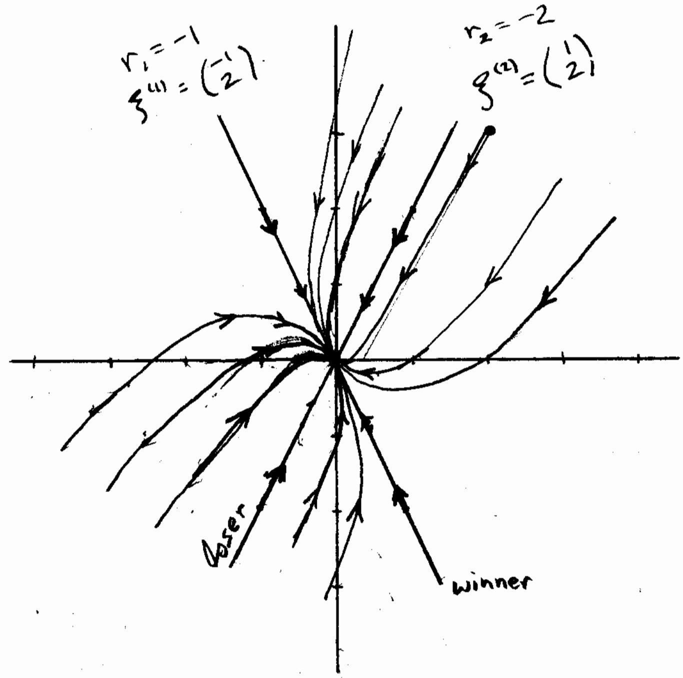
Then $AT = TR$,
So $A = TRT^{-1}$!

For an arbitrary 2×2 matrix,
$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
(Verify this.)

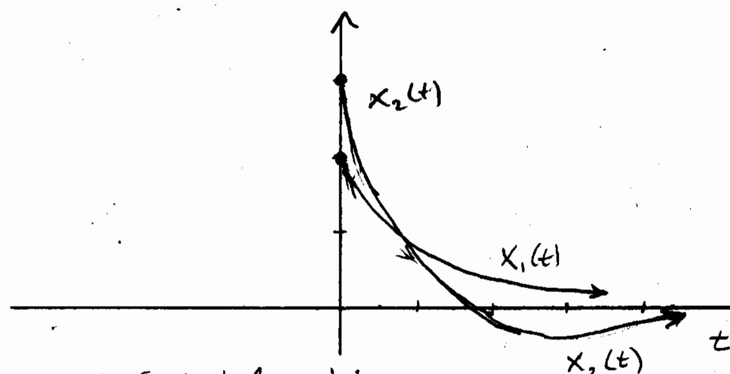
$$\text{So } T^{-1} = \frac{-1}{4}\begin{bmatrix} 2 & -1 \\ -2 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\text{So } A = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\left(\frac{1}{4}\right)\begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$
$$= \left(\frac{1}{4}\right)\begin{bmatrix} 1 & -2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$
$$= \left(\frac{1}{4}\right)\begin{bmatrix} -6 & -1 \\ -4 & -6 \end{bmatrix}$$

(But you only need the eigenvalues & eigenvectors to draw rough sketches.)



(c) For the trajectory through $(2,3)$, sketch the graphs of x_1 versus t and of x_2 versus t on the same set of axes.



Satisfy initial conditions:
$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$c_1 = \frac{-1}{4}, \quad c_2 = \frac{7}{4}$$

$$\text{So } x(t) = \frac{-1}{4}\begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + \frac{7}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$$

i.e. $x_1(t) = \frac{1}{4}\left[e^{-t} + 7e^{-2t}\right]$
$x_2(t) = \frac{1}{4}\left[-2e^{-t} + 14e^{-2t}\right]$

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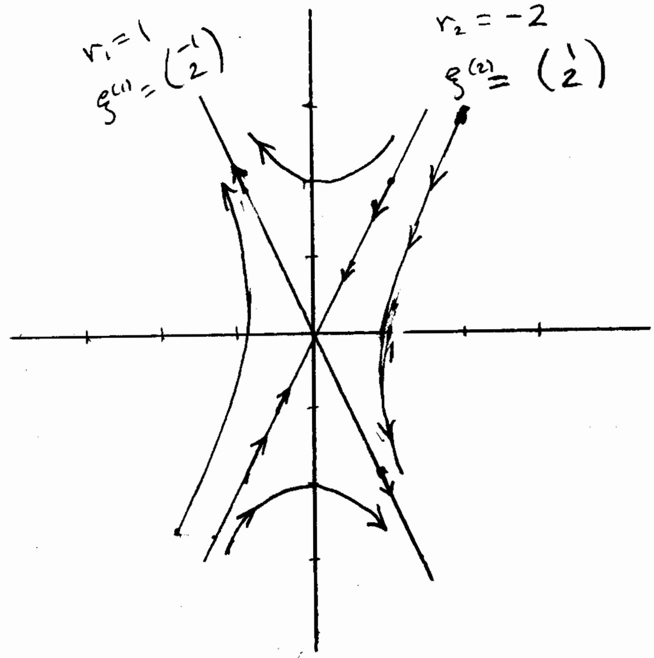
(25) Let $x' = Ax$
where the eigenvalues and eigenvectors of A are:

$\boxed{r_1 = 1}$ $\xi^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$r_2 = -2$ $\xi^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(a) Sketch phase portrait
(b) Sketch trajectory through (2,3)



The general solution is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$$

Fitting to initial condition:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

So $c_1 = \frac{-1}{4}$, $c_2 = \frac{7}{4}$

So $x_1(t) = \frac{1}{4}\left[e^{t} + 7e^{-2t}\right]$

$x_2(t) = \frac{1}{4}\left[-2e^{t} + 14e^{-2t}\right]$

This problem is identical to problem 24 except that $r_1 = 1$, not -1.

Note: $A = TRT^{-1}$ now becomes

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \left(\frac{1}{4}\right) \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \left(\frac{1}{4}\right) \begin{bmatrix} -1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \left(\frac{1}{4}\right) \begin{bmatrix} -2 & -3 \\ -12 & -2 \end{bmatrix}$$