1. Solve the initial value problem

(20 pts)

$$\frac{dy}{dt} = (\sin t)y + e^{-\cos t}, \quad y(0) = 1.$$

ANS.

(5 pts) $\frac{dy}{dt} - (\sin t)y = e^{-\cos t}$

$$\mu(t) = e^{\int_{\pi/2}^{t} -\sin s\, ds} = e^{\cos t}$$

(5 pts) $\frac{d}{dt}\{e^{\cos t} \cdot y\} = e^{\cos t} \cdot e^{-\cos t} = 1$

$$\int_0^t \frac{d}{dt}\{e^{\cos s} \cdot y\}\, dt = \int_0^t 1 \cdot dt = t$$

(5 pts) $e^{\cos t} y(t) - e^1 = t$

$$\Rightarrow e^{\cos t} \cdot y(t) = t + e$$

$$\Rightarrow y(t) = e^{-\cos t} \cdot (t + e) \quad \leftarrow (5 \text{ pts})$$

2. Solve the initial value problem

(25 pts)

$$y'' + 2y' + y = \frac{e^{-t}}{(t+1)^2} + \cos t, \quad y(0) = 0,\ y'(0) = 0.$$

ANS.

$y(t) = \frac{1}{2} t e^{-t} + \frac{1}{2}\sin t - e^{-t}\ln(t+1)$

First, homog. eqn: (5 pts)

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0, \quad r = -1$

$y_1(t) = e^{-t}, \quad y_2(t) = te^{-t}$

Second, solve particular solution

Method 1 break up RHS and solve

① $y'' + 2y' + y = \cos t$

② $y'' + 2y' + y = \frac{e^{-t}}{(t+1)^2}$

① (5 pts) $Y_{p_1} = A\cos t + B\sin t$

$Y_{p_1}' = -A\sin t + B\cos t$

$Y_{p_1}'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t + 2(-A\sin t + B\cos t) + A\cos t + B\sin t = \cos t$

$\cos t: \quad -A + 2B + A = 1 \Rightarrow B = \frac{1}{2}$

$\sin t: \quad -B - 2A + B = 0 \Rightarrow A = 0$

$Y_{p_1} = \frac{1}{2}\sin t.$

② (8 pts) Variation of parameters

$Y_{p_2} = u_1(t)y_1(t) + u_2(t)y_2(t)$

$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-t}(e^{-t} - te^{-t}) + t \ldots = \boxed{e^{-2t}}$

$u_1 = -\int \frac{te^{-t} \cdot \frac{e^{-t}}{(t+1)^2}}{e^{-2t}} dt = -\int \frac{t}{(t+1)^2} dt$

let $u = t+1$, $du = dt$

$= -\int \frac{u-1}{u^2} du = -\int \frac{1}{u} du + \int \frac{1}{u^2} du$

$= -\ln u - \frac{1}{u} \cdot (+A_1)$

$= -\ln(t+1) - \frac{1}{t+1} (+A_1)$

$u_2 = \int \frac{e^{-t} \cdot \frac{e^{-t}}{(t+1)^2}}{e^{-2t}} dt = \int \frac{1}{(t+1)^2} dt = -\frac{1}{t+1}($

$Y_{p_2} = e^{-t}\left(-\ln(t+1) - \frac{1}{t+1}\right) + te^{-t}\left(-\frac{1}{t+1}\right.$

$= \boxed{-e^{-t}\ln(t+1) - \frac{e^{-t}}{t+1} - \frac{te^{-t}}{t+1}}$

$= \boxed{-e^{-t}\ln(t+1) - e^{-t}}$ both are OK,

Take $Y_{p_2} = -e^{-t}\ln(t+1)$

since $e^{-t} = y_1(t)$.

Third, general solution (2 pts)

$$y(t) = C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2}\sin t - e^{-t}\ln(t+1).$$

Finally, solve for C_1, C_2 using initial value (5 pts)

$$y'(t) = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t} + \frac{1}{2}\cos t + e^{-t}\ln(t+1) - e^{-t}\cdot\frac{1}{t+1}$$

$$y(0)=0 \Rightarrow \quad 0 = C_1$$

$$y'(0)=0 \Rightarrow \quad 0 = C_2 + \frac{1}{2} - 1 \Rightarrow C_2 = \frac{1}{2}$$

$$y(t) = \frac{1}{2} t e^{-t} + \frac{1}{2}\sin t - e^{-t}\ln(t+1).$$

If you are using $Y_p = -e^{-t}\ln(t+1) - e^{-t}$, $C_1 = 1$, $C_2 = \frac{1}{2}$.

Method 2 for Second part (solving for particular solution) (not recommended)

(3 pts) $W = e^{-2t}$

$$\int e^t\cos t\,dt = e^t\cos t + \int e^t\sin t\,dt = e^t\cos t + e^t\sin t - \int e^t\cos t\,dt$$
$$\Rightarrow 2\int e^t\cos t\,dt = e^t(\cos t + \sin t)$$
$$\Rightarrow \int e^t\cos t\,dt = \frac{e^t(\cos t + \sin t)}{2}$$

$$u_1 = -\int \frac{te^{-t}\frac{e^{-t}}{(t+1)^2} + te^{-t}\cos t}{e^{-2t}}\,dt = -\int \frac{t}{(t+1)^2}\,dt - \int te^{t}\cos t\,dt$$

By Parts

$$= -\ln(t+1) - \frac{1}{t+1} - \frac{e^t}{2}(\sin t + \cos t)\cdot t + \int \frac{e^t}{2}(\sin t + \cos t)\,dt$$

$$= -\ln(t+1) - \frac{1}{t+1} - \frac{e^t}{2}(\sin t + \cos t)t + \frac{1}{4}e^t(\sin t + \cancel{\cos t}) + \frac{1}{4}e^t(\sin t - \cancel{\cos t})$$

$$= -\ln(t+1) - \frac{1}{t+1} - \frac{te^t}{2}(\sin t + \cos t) + \frac{1}{2}e^t\sin t$$

$$u_2 = \int \frac{e^{-t}\cdot\frac{e^{-t}}{(t+1)^2} + e^{-t}\cos t}{e^{-2t}}\,dt = \int \frac{1}{(t+1)^2}\,dt + \int e^t\cos t\,dt = -\frac{1}{t+1} + \frac{e^t}{2}(\sin t + \cos t)$$

$$Y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$= -e^{-t}\ln(t+1) - \frac{e^{-t}}{t+1} - \frac{t}{2}\cancel{e^{-t}e^{t}(\sin t + \cos t)} + \frac{1}{2}e^{-t}\cdot e^t\sin t - \frac{te^{-t}}{t+1} + \frac{t}{2}\cancel{(\sin t + \cos t)}$$

$$= -e^{-t}\ln(t+1) - e^{-t} + \frac{1}{2}\sin t$$

Take $Y_p = -e^{-t}\ln(t+1) + \frac{1}{2}\sin t$. Continue with Third part.
