1. Use Laplace transforms to solve

$$y'' + 9y = g(t),\ \ y(0) = 0,\ y'(0) = 1,$$

where

$$g(t) = \begin{cases} \frac{t}{6}, & 0 \le t < 12 \\ 2, & 12 < t \end{cases}$$

$$g(t) = u_{12}(t) \times 2 + \frac{t}{6} - \frac{t}{6} u_{12}(t)$$

$$= \frac{t}{6} + u_{12}(t)\left[2 - \frac{t}{6}\right]$$

$$= \frac{t}{6} - u_{12}(t) \cdot \frac{t-12}{6}$$

8 pts

$Y(s) = \mathcal{L}\{y(t)\}$

$$s^2 Y(s) - s y(0) - y'(0) + 9Y(s) = \frac{1}{6s^2} - \frac{e^{-12s}}{6s^2}$$

$$(s^2+9)Y(s) = 1 + \frac{1}{6s^2} - \frac{e^{-12s}}{6s^2}$$

$$Y(s) = \frac{1}{s^2+9} + \frac{1}{6} \cdot \frac{1}{s^2(s^2+9)} - \frac{e^{-12s}}{6s^2(s^2+9)}$$

$$\frac{1}{s^2(s^2+9)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+9}$$

$$1 = (As+B)(s^2+9) + (Cs+D)s^2$$

s^3: $0 = A + C$

s^2: $0 = B + D$

s: $0 = 9A$

Const: $1 = 9B$

$\Rightarrow \begin{cases} C = 0 \\ D = -1/9 \\ A = 0 \\ B = 1/9 \end{cases}$

5 pts

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{9}\cdot\frac{1}{s^2} - \frac{1}{9}\frac{1}{s^2+9}\right\}$$

$$= \frac{1}{9}t - \frac{1}{9}\cdot\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{1}{9}t - \frac{1}{27}\sin 3t$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{e^{-12s}}{s^2(s^2+9)}\right\}$$

$$= \frac{1}{3}\sin 3t + \frac{1}{6}\cdot\left[\frac{1}{9}t - \frac{1}{27}\sin 3t\right] - \frac{1}{6}\left[\frac{1}{9}(t-12) - \frac{1}{27}\sin 3(t-12)\right]u_{12}(t)$$

7 pts

2. Solve using Laplace transforms

$$2y'' + y' + 4y = \delta(t - \frac{\pi}{6}) \sin t,$$

$$y(0) = 0, y'(0) = 0.$$

($\sin \frac{\pi}{6} = \frac{1}{2}, \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$) and δ is the unit impulse function.

ANS.

(30 pts)

$$\mathcal{L}[2y''+y'+4y] = (2s^2+s+4)Y(s) -2s\,y(0) -2s\,y'(0) - y(0) = (2s^2+s+4)Y(s) \qquad (6)$$

$$\mathcal{L}(\delta(t-\tfrac{\pi}{6})\sin t) = \int_0^\infty \delta(t-\tfrac{\pi}{6}) \sin t\, e^{-st} dt = \sin\tfrac{\pi}{6}\, e^{-\frac{\pi}{6}s} = \tfrac{1}{2} e^{-\frac{\pi}{6}s} \qquad (7)$$

$$\Rightarrow Y(s) = \frac{1}{4} \frac{1}{s^2+\frac{1}{2}s+2} e^{-\frac{\pi}{6}s} = \frac{1}{4} \frac{1}{(s+\frac{1}{4})^2+2-\frac{1}{16}} e^{-\frac{\pi}{6}s} = \frac{1}{4} \frac{1}{(s+\frac{1}{4})^2+\frac{31}{16}} e^{-\frac{\pi}{6}s} \qquad \cancel{0}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+\frac{1}{4})^2+(\frac{\sqrt{31}}{4})^2}\right) = \frac{4}{\sqrt{31}} e^{-\frac{1}{4}t} \sin\left(\frac{\sqrt{31}}{4} t\right) \qquad (10)$$

$$\Rightarrow y(t) = \frac{1}{\sqrt{31}} e^{-\frac{1}{4}(t-\frac{\pi}{6})} \sin\left(\frac{\sqrt{31}}{4}(t-\tfrac{\pi}{6})\right) u_{\pi/6}(t) \qquad (7)$$

3. Find the Laplace transform of the given function

$$h(t) \overset{\text{def}}{=} \int_0^t \sin(t-\tau)\cos\tau\, d\tau.$$

ANS. $\dfrac{s}{(s^2+1)^2}$

(10 pts)

Let $f(t) = \sin t$

Let $g(t) = \cos t$

$h(t) = (f * g)(t)$ (convolution product)

so $H(s) = F(s)\,G(s)$

$$= \frac{1}{s^2+1}\,\frac{s}{s^2+1}$$

4. (a) Use the eigenvalue – eigenvector method to find the general solution of

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}}_{\text{Call } A} \mathbf{x}.$$

ANS.

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

(20 pts)

(10) Eigenvalues

$\det(A - Ir)$

$= \begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix}$

$= (1-r)(-2-r) - 4$

$= r^2 + r - 6$

$= (r-2)(r+3)$

(10) Eigenvectors

$r_1 = -3$: $\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Choose $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$r_2 = 2$: $\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Choose $\underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) Use the result of part (a) to solve the initial value problem with

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

ANS.

(1) $$\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

(10 pts)

Need $c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3 \cdot 0} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2 \cdot 0} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

i.e. $c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

(5) i.e. $\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

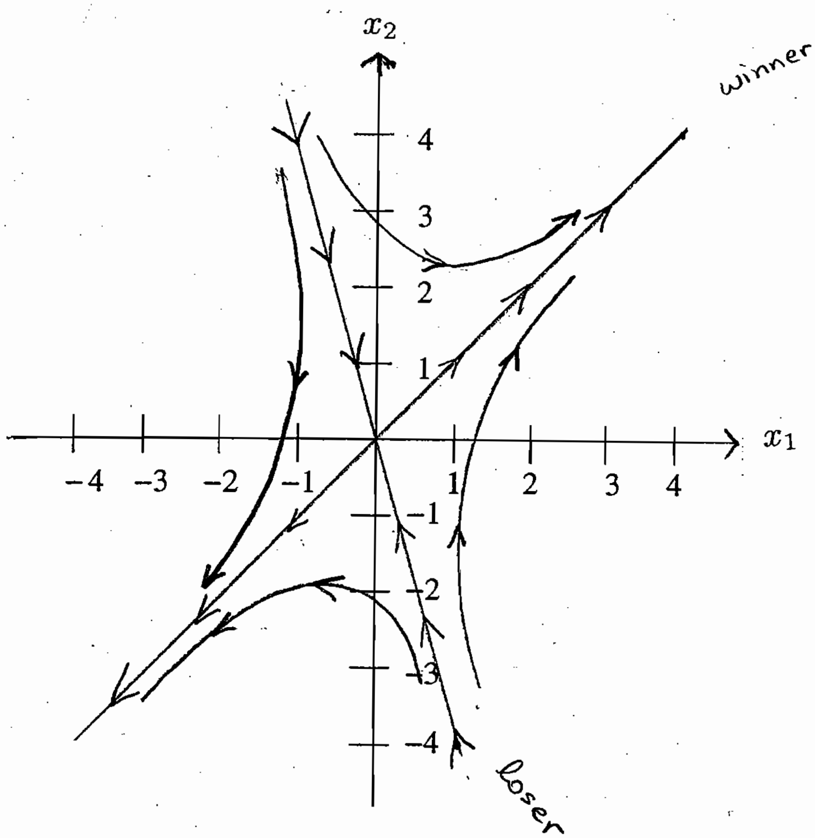
(row 2 - row 1) $\begin{bmatrix} 1 & 1 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

[(row 2) / 5] $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(row 1 - row 2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\left.\begin{matrix} c_1 = 1 \\ c_2 = 1 \end{matrix}\right\}$ (4)

(c) Use the result of part (a) to plot a few trajectories in the x_1, x_2 plane. Use arrows to denote increasing t.



(10 pts)