1. Solve the initial value problem via the eigenvalue – eigenvector method:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}.$$

ANS.

$$\underline{x}(t) = e^{4t}\begin{pmatrix} \cos 3t - \sin 3t \\ \cos 3t + \sin 3t \end{pmatrix}$$

(20 pts)

eigenvalues

$0 = \det(A - Ir)$

$= \det\begin{bmatrix} 4-r & -3 \\ 3 & 4-r \end{bmatrix}$

$= r^2 - (4+4)r + (4\cdot 4 - 3\cdot(-3))$

$(= r^2 - (\mathrm{tr} A) r + (\det A))$

$= r^2 - 8r + 25$

$= (r-4)^2 + 9$

$= (r-4)^2 - (3i)^2$

$= (r-4-3i)(r-4+3i)$

eigenvectors

$r_\pm = 4 \pm 3i$

Need $(A - Ir_\pm)\underline{\xi}^{(\pm)} = \underline{0}$.

$$\begin{bmatrix} \mp 3i & -3 \\ 3 & \mp 3i \end{bmatrix}\underbrace{\begin{pmatrix} \pm i \\ 1 \end{pmatrix}}_{\text{Call } \underline{\xi}^{(\pm)}} = \underline{0}$$

Complex solution

$\underline{x}^{(+)}(t) = \xi^{(+)} e^{r_+ t}$

$= \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(4+3i)t}$

Determine real and imaginary parts of complex solution to obtain a fundamental pair of real solutions.

$\underline{x}^{(+)}(t) = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] e^{4t}(\cos 3t + i \sin 3t)$

$= e^{4t}\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + i\, e^{4t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$

General solution

$\underline{x}(t) = e^{4t}\left(\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix} c_1 + \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} c_2\right)$

Satisfy initial conditions

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} c_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$c_1 = 1$

$c_2 = 1$

$\boxed{2a}$ $\begin{cases} \underline{x}' = A\underline{x}, & \underline{x}(0) = \begin{pmatrix}1\\1\end{pmatrix} \\ A = \begin{bmatrix}3 & 7\\1 & -3\end{bmatrix} \end{cases}$

Guess $\underline{x}(t) = \underline{\xi} e^{rt}$

Need $\underline{x}' = A\underline{x}$

$$\underline{\xi} r \cancel{e^{rt}} = A\underline{\xi}\cancel{e^{rt}}$$

$$(A - Ir)\underline{\xi} = 0$$

(5) eigenvalues

$$0 = \det(A - Ir)$$

$$= \det\begin{bmatrix}3-r & 7\\1 & -3-r\end{bmatrix}$$

$$= r^2 - (3+-3)r + (3\cdot(-3) - 1\cdot 7)$$

$$= r^2 - 16$$

$$= (r-4)(r+4)$$

(5) eigenvectors

$$r_1 = -4: \quad \underbrace{\begin{bmatrix}7 & 7\\1 & 1\end{bmatrix}}_{A - Ir_1}\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\underline{\xi}_1} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$r_2 = 4 \quad \underbrace{\begin{bmatrix}-1 & 7\\1 & -7\end{bmatrix}}_{A - I4}\begin{pmatrix}7\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

general homogeneous solution

$$\underline{x}^{(h)}(t) = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + c_2\begin{pmatrix}7\\1\end{pmatrix}e^{4t}$$

satisfy initial conditions

$$\begin{pmatrix}1\\1\end{pmatrix} = c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}7\\1\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{bmatrix}1 & 7\\-1 & 1\end{bmatrix}^{-1}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{8}\begin{bmatrix}1 & -7\\1 & 1\end{bmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}-3\\1\end{pmatrix}$$

(10) (1) $\boxed{\underline{x}(t) = \frac{-3}{4}\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + \frac{1}{4}\begin{pmatrix}7\\1\end{pmatrix}e^{4t}}$

$\boxed{2b}$ $\begin{cases} \underline{x}' = A\underline{x} + \underline{g}(t), & \underline{x}(0) = \begin{pmatrix}1\\1\end{pmatrix} \\ A = \begin{bmatrix}3 & 7\\1 & -3\end{bmatrix}, & \underline{g}(t) = e^t\begin{pmatrix}-5\\3\end{pmatrix} \end{cases}$

Need particular solution.

(10) Guess $\underline{x}^{(p)}(t) = \underline{a}e^t$ (Method of Undetermined Coefficients)

Need $\underline{x}^{(p)\prime} = A\underline{x}^{(p)} + \underline{g}$

$$\underline{a}\cancel{e^t} = A\underline{a}\cancel{e^t} + \begin{pmatrix}-5\\3\end{pmatrix}\cancel{e^t}$$

$$(A - I)\underline{a} = \begin{pmatrix}5\\-3\end{pmatrix}$$

$$\underline{a} = (A-I)^{-1}\begin{pmatrix}5\\-3\end{pmatrix}$$

$$= \begin{bmatrix}2 & 7\\1 & -4\end{bmatrix}^{-1}\begin{pmatrix}5\\-3\end{pmatrix}$$

$$= \frac{1}{15}\begin{bmatrix}4 & 7\\1 & -2\end{bmatrix}\begin{pmatrix}5\\-3\end{pmatrix}$$

$$= \frac{1}{15}\begin{pmatrix}-1\\11\end{pmatrix}$$

General nonhomogeneous solution

$$\underline{x}(t) = \underline{x}^{(h)}(t) + \underline{x}^{(p)}(t)$$

(2) $$= c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + c_2\begin{pmatrix}7\\1\end{pmatrix}e^{4t} + \frac{1}{15}\begin{pmatrix}-1\\11\end{pmatrix}e^t$$

satisfy initial conditions

(2) $$\begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix}1 & 7\\-1 & 1\end{bmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} + \frac{1}{15}\begin{pmatrix}-1\\11\end{pmatrix}$$

(5) $$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{bmatrix}1 & 7\\-1 & 1\end{bmatrix}^{-1}\left\{\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{15}\begin{pmatrix}-1\\11\end{pmatrix}\right\}$$

$$= \frac{1}{8}\begin{bmatrix}1 & -7\\1 & 1\end{bmatrix}\frac{1}{15}\begin{pmatrix}16\\4\end{pmatrix}$$

$$= \frac{1}{15}\frac{1}{2}\begin{bmatrix}1 & -7\\1 & 1\end{bmatrix}\begin{pmatrix}4\\1\end{pmatrix}$$

$$= \frac{1}{15}\frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix}$$

$$= \begin{pmatrix}-1/10\\1/6\end{pmatrix}$$

(10) (1) $\boxed{\underline{x}(t) = \frac{-1}{10}\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + \frac{1}{6}\begin{pmatrix}7\\1\end{pmatrix}e^{4t} + \frac{e^t}{15}\begin{pmatrix}-1\\11\end{pmatrix}}$

$\boxed{2b}$ $\underline{x}' = A\underline{x} + g(t)$, $\underline{x}(0) = \begin{pmatrix}1\\1\end{pmatrix}$

$A = \begin{bmatrix}3 & 7\\1 & -3\end{bmatrix}$, $g(t) = e^t\begin{pmatrix}-5\\3\end{pmatrix}$

Homogeneous solution:

$$x^{(h)}(t) = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + c_2\begin{pmatrix}7\\1\end{pmatrix}e^{4t}$$

$$= \underbrace{\begin{bmatrix}e^{-4t} & 7e^{4t}\\-e^{-4t} & e^{4t}\end{bmatrix}}_{\text{Call } \Psi(t)\text{, a fundamental matrix.}}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

(2)

Note $\Psi' = A\Psi$

Need particular solution:

Method 2: Variation of parameters

Guess $\underline{x} = \Psi(t)\,u(t)$

Need $\underline{x}' = A\underline{x} + g$

$$\cancel{\Psi' u} + \Psi u' = \cancel{A\Psi u} + g$$

$$u = \int \Psi^{-1} g = \underline{c} + \int_0^t \Psi^{-1} g$$

$$\underline{x} = \Psi u = \Psi\left(\underline{c} + \int_0^t \Psi^{-1} g\right)$$

Application:

Need Ψ^{-1}!

$\det \Psi = 8$

(2) $\Psi^{-1} = \frac{1}{8}\begin{bmatrix}e^{4t} & -7e^{4t}\\e^{-4t} & e^{-4t}\end{bmatrix}$

$g = e^t\begin{pmatrix}-5\\3\end{pmatrix}$

(2) $u' = \Psi^{-1} g = \frac{1}{8}\begin{pmatrix}-5e^{5t} - 21e^{5t}\\-5e^{-3t} + 3e^{-3t}\end{pmatrix}$

$$= \frac{1}{8}\begin{pmatrix}-26e^{5t}\\-2e^{-3t}\end{pmatrix}$$

$$u' = \frac{1}{4}\begin{pmatrix}-13e^{5t}\\-1e^{-3t}\end{pmatrix}$$

(2) $u = \int \Psi^{-1} g = \frac{1}{4}\begin{pmatrix}-\frac{13}{5}e^{5t}\\\frac{1}{3}e^{-3t}\end{pmatrix} + \underline{c}$

(2) $\underline{x} = \Psi u = \begin{bmatrix}e^{-4t} & 7e^{4t}\\-e^{-4t} & e^{4t}\end{bmatrix}\left(\frac{1}{4}\begin{pmatrix}-\frac{13}{5}e^{5t}\\\frac{1}{3}e^{-3t}\end{pmatrix} + \underline{c}\right)$

$$= \frac{1}{4}\begin{pmatrix}(-4/15)e^t\\(44/15)e^t\end{pmatrix} + \Psi\underline{c}$$

$$= \begin{pmatrix}-1/15\, e^t\\11/15\, e^t\end{pmatrix} + \Psi\underline{c}$$

Method 3: diagonalization

Let $T = \begin{bmatrix}1 & 7\\-1 & 1\end{bmatrix}$ = matrix of eigenvectors

Let $D = \begin{bmatrix}-4 & \\ & 4\end{bmatrix}$ = matrix of eigenvalues

$AT = TD$. So $A = TDT^{-1}$

$$\underline{x}' = A\underline{x} + g$$

$$\underline{x}' = TDT^{-1}\underline{x} + g$$

$$\underbrace{T^{-1}\underline{x}'}_{\underline{y}'} = D\,\underbrace{T^{-1}\underline{x}}_{\text{Call } \underline{y}} + T^{-1}g$$

(2) $T^{-1} = \frac{1}{8}\begin{bmatrix}1 & -7\\1 & 1\end{bmatrix}$

$g = e^t\begin{pmatrix}-5\\3\end{pmatrix}$

(2) $T^{-1}g = e^t\left(\frac{1}{8}\right)\begin{pmatrix}-26\\-2\end{pmatrix} = e^t\,\frac{1}{4}\begin{pmatrix}-13\\-1\end{pmatrix}$

$$= \begin{pmatrix}(-13/4)e^t\\(-1/4)e^t\end{pmatrix}$$

$$\begin{pmatrix}y_1'\\y_2'\end{pmatrix} = \begin{bmatrix}-4 & \\ & 4\end{bmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} + T^{-1}g$$

(2) $\begin{cases} y_1' = -4y_1 - 13/4\, e^t \\ y_2' = 4y_1 - 1/4\, e^t \end{cases}$

Can solve these equations by many methods, such as undetermined coefficients.

Merely need particular solutions.

Guess $y_1(t) = ae^t$. Get:

(2) $\begin{Bmatrix} y_1(t) = -\frac{13}{20}e^t \\ y_2(t) = \frac{1}{12}e^t \end{Bmatrix}$ $\underline{y} = \frac{1}{4}\begin{pmatrix}-13/5\\1/3\end{pmatrix}e^t$

(2) $\underline{x} = T\underline{y} = \frac{1}{4}\begin{bmatrix}1 & 7\\-1 & 1\end{bmatrix}\begin{pmatrix}-13/5\\1/3\end{pmatrix}e^t$

$$= \begin{pmatrix}-1/15\\11/15\end{pmatrix}e^t$$

3) $\frac{dx}{dt} = 3x - xy - 2x^2$

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#3

$\frac{dy}{dt} = y - y^2 - xy$

a) Find all critical points

$0 = 3x - xy - 2x^2$
$= x(3 - y - 2x)$
$x = 0, \quad y = 3 - 2x$

$0 = y - y^2 - xy$
$= y(1 - y - x)$
$y = 0, \quad y = 1 - x$

+5 solving $\frac{dx}{dt}=0, \frac{dy}{dt}=0$
+1 $(0,0), (0,1), (\frac{3}{2},0)$
+2 $(2,-1)$

critical points: $\boxed{(0,0), \quad (0,1) \quad (\frac{3}{2},0), \quad (2,-1)}$

$0 = 3 - 2x$
$x = \frac{3}{2}$

$1 - x = 3 - 2x$
$x = 2$
$y = 1 - 2 = -1$

b) Find the corresponding linear system near each critical point.

+6 partials
+4 system

$F(x,y) = 3x - xy - 2x^2$ $\quad F_x(x,y) = 3 - y - 4x$ $\quad F_y(x,y) = -x$

$G(x,y) = y - y^2 - xy$ $\quad G_x(x,y) = -y$ $\quad G_y(x,y) = 1 - 2y - x$

at $(0,0)$
$$\underline{u}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \underline{u}$$

at $(0,1)$
$$\underline{u}' = \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} \underline{u}$$

at $(2,-1)$
$$\underline{u}' = \begin{pmatrix} -4 & -2 \\ 1 & 1 \end{pmatrix} \underline{u}$$

at $(\frac{3}{2},0)$:
$$\underline{u}' = \begin{pmatrix} -3 & -3/2 \\ 0 & -1/2 \end{pmatrix}$$

(c) Find the eigenvalues of each linear system and classify the critical points.

at $(0,0)$: $0 = \det\begin{pmatrix} 3-r & 0 \\ 0 & 1-r \end{pmatrix} = (3-r)(1-r)$

$r = 1, 3$ unstable node

at $(0,1)$: $0 = \det\begin{pmatrix} 2-r & 0 \\ -1 & -1-r \end{pmatrix} = (2-r)(-1-r)$

$r = 2, -1$ unstable saddle

at $(2,-1)$ $0 = \det\begin{pmatrix} -4-r & -2 \\ 1 & 1-r \end{pmatrix} = (-4-r)(1-r)+2 = r^2 - r + 4r - 4 + 2$

$= r^2 + 3r - 2$

$r = \frac{-3 \pm \sqrt{9+8}}{2}$ $\quad r = \frac{-3+\sqrt{17}}{2}, \frac{-3-\sqrt{17}}{2}$ $\quad \sqrt{17} > 3$, so

unstable saddle

at $\left(\frac{3}{2}, 0\right)$ $0 = \det\begin{pmatrix} -3-r & -3/2 \\ 0 & -\frac{1}{2}-r \end{pmatrix} = (-3-r)(-\tfrac{1}{2}-r)$

$r = -3, -\frac{1}{2}$ asymptotically stable node

+2 each ev.

+2 "concept."

d) Sketch a phase portrait for the nonlinear system.

near $(0,0)$: $\underline{u}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \underline{u}$, $r = 3, 1$,

$r=3$ $\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\xi_2 = 0$ $\quad \underline{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$r=1$ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2\xi_1 = 0$ $\quad \underline{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

near $(0,1)$ $\underline{u}' = \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} \underline{u}$

$r = 2, -1$

$r=2$: $\begin{pmatrix} 0 & 0 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\xi_1 = -3\xi_2$ $\quad \underline{\xi}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

+4 e.v.

+6 graph

$r=-1$: $\begin{pmatrix} 3 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3\xi_1 = 0$ $\quad \underline{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

near $(2,-1)$ $\underline{u}' = \begin{pmatrix} -4 & -2 \\ 1 & 1 \end{pmatrix} \underline{u}$

$r = \frac{-3+\sqrt{17}}{2}, \frac{-3-\sqrt{17}}{2}$

$r = \frac{-3\pm\sqrt{17}}{2}$ $\quad \begin{pmatrix} -4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \left(\frac{-3\pm\sqrt{17}}{2}\right)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

$\xi_1 + \xi_2 = \frac{-3\pm\sqrt{17}}{2}\xi_2$, $\quad \xi_1 = \frac{-5\pm\sqrt{17}}{2}\xi_2$

$\frac{-3+\sqrt{17}}{2}$ $\quad \underline{\xi}^{(1)} = \begin{pmatrix} -5+\sqrt{17} \\ 2 \end{pmatrix}$ $\quad \underline{\xi}^{(2)} = \begin{pmatrix} -5-\sqrt{17} \\ 2 \end{pmatrix}$

$\frac{-3-\sqrt{17}}{2}$

near $\left(\frac{3}{2}, 0\right)$ $\underline{u} = \begin{pmatrix} -3 & -3/2 \\ 0 & -1/2 \end{pmatrix} \underline{u}$

$r = -3, -\frac{1}{2}$

$r=-3$ $\quad \begin{pmatrix} 0 & -3/2 \\ 0 & 5/2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\xi_2 = 0$ $\quad \underline{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$r = -\frac{1}{2}$ $\quad \begin{pmatrix} -5/2 & -3/2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-\frac{5}{2}\xi_1 = \frac{3}{2}\xi_2$

$-5\xi_1 = 3\xi_2$

$\underline{\xi}^{(2)} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$