Sec. 7.8 p. 428 #1,2,5 Andy Raich

1) $\vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\vec{x}$ try $\vec{x} = \vec{\xi} e^{rt}$. Then $\vec{x}' = \vec{\xi} r e^{rt}$ and

and $\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$. Then $(3-r)(-1-r)+4=0$

$$r^2-2r+1=0$$
$$(r-1)^2=0 \Rightarrow r=1.$$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$, so $\xi_1 = 2\xi_2$, so $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$ is a soln.

Next, try $\vec{x} = \vec{\xi} te^t + \vec{\eta} e^t$. Then $\vec{x}' = \vec{\xi} te^t + (\vec{\xi} + \vec{\eta})e^t$ which means

$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\vec{\xi} = \vec{\xi}$ and $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\vec{\eta} = \vec{\xi} + \vec{\eta}$. Take $\vec{\xi} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Then $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so $\eta_1 - 2\eta_2 = 1$. Take $\eta_1 = 1$, $\eta_2 = 0$. Then

$$\boxed{\vec{x} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t\right)}$$

2) $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\vec{x}$. try $\vec{x} = \vec{\xi} e^{rt}$. Then $\vec{x}' = \vec{\xi} r e^{rt}$ and

$\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$. Then $(4-r)(-4-r)+16=0$

$$-16-4r+4r+r^2+16=0$$
$$r=0$$

and $4\xi_1 - 2\xi_2 = 0$

$\xi_2 = 2\xi_1$

So $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a soln.

Next, try $\vec{x} = \vec{\xi} t + \vec{\eta}$. Then $\vec{x}' = \vec{\xi}$, so

$\vec{\xi} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}(\vec{\xi} t + \vec{\eta})$. As before, we can take $\vec{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then

$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $4\eta_1 - 2\eta_2 = 1$, so if $\eta_1 = 0$, $\eta_2 = -\frac{1}{2}$ and

$$\boxed{\vec{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right)}$$

5) $\vec{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \vec{x}$. Try $\vec{x} = \vec{\xi} e^{rt}$ so $\vec{x}' = r\vec{\xi} e^{rt}$, so we need

$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0$ which means: $(1-r)((1-r)^2-1) - 2((1-r)+1) = 0$

$(1-r)^3 - (1-r) - 2(1-r) - 2 = 0$

$(1-r)^3 - 3(1-r) - 2 = 0$

Let $s = 1-r$. Then $s^3 - 3s - 2 = 0$

try $s=-1$: $-1+3-2=0$ ✓.

$$
\begin{array}{r}
s^2 - s - 2 \\
s+1 \overline{\smash{)}\, s^3 + 0s^2 - 3s - 2} \\
-(s^3 + s^2) \\
-s^2 - 3s \\
-(-s^2 - s) \\
-2s - 2 \\
-2s - 2 \\
0
\end{array}
$$

$s^3 - 3s - 2 = (s+1)(s^2 - s - 2) = (s+1)^2(s-2)$

$s = -1, 2$, so $r = 1-s = 2, -1$

with a double root at $r=2$

try $\vec{x} = \vec{\xi} e^{-t}$. Then $\vec{x}' = -\vec{\xi} e^{-t}$

and $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\xi_2 = 2\xi_3$

$2\xi_1 + \xi_2 + \xi_3 = 0$, so $2\xi_1 = -3\xi_3$

$\xi_1 = -\frac{3}{2}\xi_3$

so $\vec{x} = \begin{pmatrix} -3/2 \\ 2 \\ 1 \end{pmatrix} e^{-t}$ is a sol'n.

try $\vec{x}(t) = \vec{\xi} t e^{2t} + \vec{\eta} e^{2t}$, so $\vec{x}' = 2t\vec{\xi} e^{2t} + (\vec{\xi} + 2\vec{\eta}) e^{2t}$, so

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0$ and $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$

$\xi_2 = -\xi_3$

$+\xi_1 = \xi_2 + \xi_3 = 0$, so $\vec{\xi} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

and

$-\eta_2 = \eta_3 - 1$

$-\eta_1 + \eta_2 + \eta_3 = 0$

$-\eta_1 - \eta_3 + 1 + \eta_3 = 0$

$\eta_1 = +1$

$\eta_3 = 1, \eta_2 = 0$

$\vec{\eta} = \begin{pmatrix} +1 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -3/2 \\ 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right)}$$