Sec. 9.3 p.511 #5-7 Andy Raich

5) $\frac{dx}{dt} = (2+x)(y-x)$ $\frac{dy}{dt} = (4-x)(y+x)$

$\frac{dx}{dt} = 0$ if $x=-2$, $x=y$ $\frac{dy}{dt}=0$ if $x=4$, $x=-y$

$\boxed{\text{critical points: } (0,0), (-2,2), (4,4)}$

b) corresponding linear systems:

$F(x,y) = 2y-2x+xy-x^2$ $G(x,y) = 4y+4x-xy-x^2$

$F_x(x,y) = -2+y-2x$ $G_x(x,y) = 4-y-2x$

$F_y(x,y) = 2+x$ $G_y(x,y) = 4-x$

at $(0,0)$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\boxed{\begin{array}{l} u' = -2u+2v \\ v' = 4u+4v \end{array}}$

at $(-2,2)$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\boxed{\begin{array}{l} u' = 4u \\ v' = 6u+6v \end{array}}$

at $(4,4)$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\boxed{\begin{array}{l} u' = -6u+6v \\ v' = -8u \end{array}}$

c) eigenvalues $\left|\begin{pmatrix} -2-\lambda & 2 \\ 4 & 4-\lambda \end{pmatrix}\right| = (-2-\lambda)(4-\lambda)-8 = -8-4\lambda+2\lambda+\lambda^2-8$

$\lambda^2-2\lambda-16=0$ $\lambda = \frac{2\pm\sqrt{4+64}}{2} = \boxed{1\pm\sqrt{17}}$ $1+\sqrt{17}>0$, $1-\sqrt{17}<0$ saddle point unstable

$\begin{vmatrix} 4-r & 0 \\ 6 & 6-r \end{vmatrix} = (4-r)(6-r)$ $\boxed{r=4,6}$ node, unstable

$\begin{vmatrix} -6-r & 6 \\ -8 & -r \end{vmatrix} = 6r+r^2+48$ $r = \frac{-6\pm\sqrt{36-4\cdot 48}}{2} = -3\pm\sqrt{9-48} = \boxed{-3\pm i\sqrt{39}}$

6) $\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 3y - xy - 2y^2$

$0 = x - x^2 - xy$
$\quad = x(1 - x - y)$
$x = 0, \quad x + y = 1$

$0 = 3y - xy - 2y^2$
$0 = y(3 - x - 2y)$
$x + 2y = 3$
$y = 0$

$\boxed{\text{critical points: } (0,0) \quad (1,0) \quad (0, 3/2) \quad (-1, 2)}$

$$\begin{array}{r} x + y = 1 \\ -\ (x + 2y = 3) \\ \hline -y = -2 \\ y = 2 \end{array}$$

b) corresponding linear system

$F(x,y) = x - x^2 - xy \qquad G(x,y) = 3y - xy - 2y^2$
$F_x(x,y) = 1 - 2x - y \qquad G_x(x,y) = -y$
$F_y(x,y) = -x \qquad G_y(x,y) = 3 - x - 4y$

at $(0,0)$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\boxed{\begin{array}{l} u' = u \\ v' = 3v \end{array}}$

at $(1,0)$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\boxed{\begin{array}{l} u' = -u - v \\ v' = 2v \end{array}}$

at $(0, 3/2)$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ -3/2 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\boxed{\begin{array}{l} u' = -\frac{1}{2}u \\ v' = -\frac{3}{2}u - 3v \end{array}}$

at $(-1,2)$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\boxed{\begin{array}{l} u' = u + v \\ v' = -2u - 4v \end{array}}$

c) eigenvalues:

$(0,0)$: $\begin{vmatrix} 1-r & 0 \\ 0 & 3-r \end{vmatrix} = (1-r)(3-r) = 0 \Rightarrow r = 1, 3$ node, unstable

$(1,0)$: $\begin{vmatrix} -1-r & -1 \\ 0 & 2-r \end{vmatrix} = (-1-r)(2-r) = 0 \Rightarrow r = -1, 2$ saddle point, unstable

$(0, 3/2)$: $\begin{vmatrix} -1/2 - r & 0 \\ -3/2 & -3-r \end{vmatrix} = (-\frac{1}{2} - r)(-3 - r) = 0 \Rightarrow r = -\frac{1}{2}, -3$ node, asymptotically stable

$(-1,2)$: $\begin{vmatrix} 1-r & 1 \\ -2 & -4-r \end{vmatrix} = (1-r)(-4-r) + 2 = 0$

$-4 - r + 4r + r^2 + 2 = 0$

$r^2 + 3r - 2 = 0$ saddle point, unstable

$r = \frac{-3 \pm \sqrt{9+8}}{2} = -\frac{3}{2} \pm \frac{\sqrt{17}}{2}$

7) $\frac{dx}{dt} = 1-y \qquad \frac{dy}{dt} = x^2-y^2$

$0 = 1-y$

$y = 1$

$0 = x^2-y^2$
$= (x-y)(x+y)$
$x = \pm y$

critical points
$(1,1), (-1,1)$

b) $F(x,y) = 1-y \qquad G(x,y) = x^2-y^2$

$F_x(x,y) = 0 \qquad G_x(x,y) = 2x$

$F_y(x,y) = -1 \qquad G_y(x,y) = -2y$

at $(1,1)$: $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\quad u' = -v$, $v' = 2u-2v$

at $(-1,1)$: $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\quad u' = -u$, $v' = -2u-2v$

c) eigenvalues

at $(1,1)$ $\begin{vmatrix} 0-r & -1 \\ 2 & -2-r \end{vmatrix} = -r(-2-r)+2 = 0$

$r^2+2r+2 = 0$

$r = \frac{-2 \pm \sqrt{4-8}}{2}$

$r = -1 \pm i$ spiral point
asymptotically stable

at $(-1,1)$: $\begin{vmatrix} -r & -1 \\ -2 & -2-r \end{vmatrix} = -r(-2-r)-2 = 0$

$r^2+2r-2 = 0$

$r = \frac{-2 \pm \sqrt{4+8}}{2}$

$r = -1 \pm \sqrt{3}$ saddle point
unstable