The Geospace Environmental Modeling (GEM) Magnetic Reconnection Challenge Problem

by E. Alec Johnson, October 23, 2008

Summary. The Geospace Environmental Modeling Magnetic Reconnection Challenge Problem (alias "GEM reconnection problem") [Birn01] is a well-studied test problem that exhibits fast reconnection in collisionless plasma. These notes specify the problem and the models we are using to solve it.

1 Plasma Models

1.1 Ideal MHD model.

The system of equations used in the ideal MHD model is

$$\partial_t \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \tilde{\mathcal{E}} \\ \mathbf{B} \end{bmatrix} + \nabla\cdot \begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + \mathbb{I}\tilde{p} - \mu_0^{-1}(\mathbf{B}\mathbf{B}) \\ \mathbf{u}(\tilde{\mathcal{E}}+\tilde{p}) - \mu_0^{-1}\mathbf{B}\mathbf{B}\cdot\mathbf{u} \\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u} \end{bmatrix} = 0.$$

Here ρ is mass density, $\mathbf{u}$ is plasma velocity, $\tilde{\mathcal{E}} = \mathcal{E} + \mu_0^{-1}B^2/2$ is total energy, $\mathcal{E} = (3/2)p + (1/2)\rho u^2$ is gas energy, $\tilde{p} = p + \mu_0^{-1}B^2/2$ is total pressure, p is gas-dynamic pressure, $\mathbf{B}$ is magnetic field, and μ_0 is magnetic permeability, usually nondimensionalized to 1.

1.2 Hall MHD model.

General MHD equations derived from the two-fluid model with isotropic pressure are:

$$\partial_t \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \tilde{\mathcal{E}} \\ \mathbf{B} \end{bmatrix} + \nabla\cdot \underbrace{\begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + \tilde{p}\mathbb{I} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B} \\ \mathbf{u}\left(\tilde{\mathcal{E}}+\tilde{p}\right) - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}\cdot\mathbf{u} \\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u} \end{bmatrix}}_{\text{hyperbolic flux}} = \nabla\cdot \underbrace{\begin{bmatrix} 0 \\ \tilde{m}_e\tilde{m}_i\mathbf{J}\mathbf{J}/\rho \\ -\tilde{\mathbf{q}} - \mu_0^{-1}\mathbf{E}'\times\mathbf{B} \\ \underline{\underline{\underline{\varepsilon}}}\cdot\mathbf{E}' \end{bmatrix}}_{\text{dispersive flux}};$$

physical solutions satisfy $\nabla\cdot\mathbf{B} = 0$. The quantities are defined as follows: $\underline{\underline{\underline{\varepsilon}}}$ is the permutation tensor, $\mathbf{B}$ is magnetic field, c is the speed of light, λ is the Debye length, $\mu_0^{-1} = c^2\lambda^2$ serves as permeability (i.e., λ^2 serves as permittivity), s denotes ion (i) or electron (e) species index, e is the charge of a proton, m_s is the particle mass of species s, $\tilde{m}_s := m_s/e$, $\tilde{m}^{-1} := \tilde{m}_i^{-1} + \tilde{m}_e^{-1}$, n

is particle density (of either species, assuming quasineutrality), ρ is density, $\mathbf{u}$ is fluid velocity, p is gas-dynamic pressure, $\tilde{p} := p + B^2/(2\mu_0)$ is the total pressure, $\mathcal{E} = (3/2)p + (1/2)u^2$ is gas-dynamic energy, $\tilde{\mathcal{E}} := \mathcal{E} + \mu_0^{-1}B^2/2$ is total energy, $\mathbf{J} \approxeq \mu_0^{-1}\nabla\times\mathbf{B}$ is the current, $\mathbf{w}_i = \tilde{m}_e\mathbf{J}/\rho \approxeq 0$ and $\mathbf{w}_e = -\tilde{m}_i\mathbf{J}/\rho \approxeq \mathbf{J}/(ne)$ are species velocities relative to $\mathbf{u}$, $\mathbf{u}_s := \mathbf{u} + \mathbf{w}_s$,

$$\tilde{\mathbf{q}} = \tilde{m}_i\tilde{m}_e\frac{\mathbf{JJ}}{\rho}\cdot(\mathbf{u} + \frac{\tilde{m}_e - \tilde{m}_i}{2}\frac{\mathbf{J}}{\rho}) + \frac{5}{2}\frac{\mathbf{J}}{\rho}(\tilde{m}_e p_i - \tilde{m}_i p_e)$$

represents the energy flux due to interspecies motion, $\mathbf{E}$ is electric field, and $\mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{B}$ is the electric field in the reference frame of the fluid, as supplied by Ohm's law:

$$\mathbf{E}' = \eta\mathbf{J} + \frac{\tilde{m}_i - \tilde{m}_e}{\rho}\mathbf{J}\times\mathbf{B} + \frac{r}{\rho}\nabla(\tilde{m}_e p_i - \tilde{m}_i p_e) + \frac{r\tilde{m}_i\tilde{m}_e}{\rho}\Big(\partial_t\mathbf{J} + \nabla\cdot\big(\mathbf{uJ} + \mathbf{Ju} + \frac{\tilde{m}_e - \tilde{m}_i}{\rho}\mathbf{JJ}\big)\Big).$$

Here η is the resistivity and r denotes the gyroradius of a typical ion. [How does the resistivity scale?] We desire to neglect the terms in the dispersive flux. Ideal MHD says that you can completely ignore it. Hall MHD says that you can neglect everything except the $\mathbf{E}'$ terms.

1.3 Five-moment model.

Collisionless two-fluid models regard the plasma as an ion fluid and an electron fluid occupying the same space region. These species fluids are coupled to the electromagnetic field, but have no direct interaction. In the five-moment model each species is modeled as a (five-moment) ideal gas.

The nondimensionalized system of equations used in the collisionless five-moment two-fluid model is

$$\partial_t\begin{bmatrix}\rho_i\\ \rho_e\\ \rho_i\mathbf{u}_i\\ \rho_e\mathbf{u}_e\\ \mathcal{E}_i\\ \mathcal{E}_e\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho_i\mathbf{u}_i\\ \rho_e\mathbf{u}_e\\ \rho_i\mathbf{u}_i\mathbf{u}_i + p_i\mathbb{I}\\ \rho_e\mathbf{u}_e\mathbf{u}_e + p_e\mathbb{I}\\ \mathbf{u}_i\big(\mathcal{E}_i + p_i\big)\\ \mathbf{u}_e\big(\mathcal{E}_e + p_e\big)\end{bmatrix} = \frac{1}{r_g}\begin{bmatrix}0\\ 0\\ \sigma_i(\mathbf{E} + \mathbf{u}_i\times\mathbf{B})\\ \sigma_e(\mathbf{E} + \mathbf{u}_e\times\mathbf{B})\\ \sigma_i\mathbf{u}_i\cdot\mathbf{E}\\ \sigma_e\mathbf{u}_e\cdot\mathbf{E}\end{bmatrix},$$

$$\partial_t\begin{bmatrix}c\mathbf{B}\\ \mathbf{E}\end{bmatrix} + c\nabla\times\begin{bmatrix}\mathbf{E}\\ -c\mathbf{B}\end{bmatrix} = \begin{bmatrix}0\\ -\mathbf{J}/\varepsilon\end{bmatrix}, \text{ and } \nabla\cdot\begin{bmatrix}c\mathbf{B}\\ \mathbf{E}\end{bmatrix} = \begin{bmatrix}0\\ \sigma/\varepsilon\end{bmatrix}.$$

These quantities are defined as follows: i and e are the ion and electron indices, ρ is density, $\mathbf{u}$ is fluid velocity, $\mathcal{E}$ is gas-dynamic energy, p is (scalar) pressure, $\mathbf{E}$ is electric field, $\mathbf{B}$ is magnetic field, $\sigma = \frac{q}{m}\rho$ is charge density, q and m are particle mass and charge ($q_i = -q_e = e$, the charge on an electron, and $m_0 := m_i + m_e$; after nondimensionalizing we may assume that $e = 1$ and $m_0 = 1$), $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_e = \sigma_i\mathbf{u}_i + \sigma_e\mathbf{u}_e$ is current, $\varepsilon = r_g\lambda_D^2$ serves as a pseudo-permittivity, $r_g := \frac{m_0u_0}{q_0B_0x_0}$ is the the ratio of the gyroradius of a typical ion to a typical length scale x_0 (m_0 and q_0 are the mass and charge of an ion, u_0 and n_0 are typical values of ion speed and number density, and B_0 is a typical magnitude of magnetic field), $\lambda_D := \sqrt{\frac{\varepsilon_0 m_0 u_0^2}{n_0 q_0^2}}$ is the ratio of a typical Debye length to the ion gyroradius, ε_0 is the permittivity of free space, and $\mathbb{I}$ is the identity tensor. The constitutive relation for the scalar pressure is given by $\mathcal{E} = (\alpha/2)p + (1/2)\rho u^2$, where α, the number of degrees of freedom per particle, equals 3.

Note that the ion pressure obeys the evolution equation $(d_t^{\mathbf{u}_i})p_i + \gamma(\nabla \cdot \mathbf{u}_i)p_i = 0$, where $(d_t^{\mathbf{u}_i}) := d_t + \mathbf{u}_i \cdot \nabla$ is the convective derivative defined by $\mathbf{u}_i$, and $\gamma = \frac{\alpha+2}{\alpha}$. Likewise for the electron pressure. These pressures should remain positive.

1.4 Ten-moment model.

In the ten-moment model, for each species we evolve a pressure tensor (with six independent components) instead of a scalar pressure.

The ten-moment model replaces the gas-dynamic energy $\mathcal{E}_s := \rho_s \langle v_s^2 \rangle / 2$ with an energy tensor $\mathbb{E}_s := \rho_s \langle \mathbf{v}_s \mathbf{v}_s \rangle$. ($\mathbf{v}_s$ is particle velocity and angle brackets denote statistical average over a small test volume). The nondimensionalized system of equations used in the collisionless ten-moment two-fluid model is

$$\partial_t \begin{bmatrix} \rho_i \\ \rho_e \\ \rho_i \mathbf{u}_i \\ \rho_e \mathbf{u}_e \\ \mathbb{E}_i \\ \mathbb{E}_e \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho_i \mathbf{u}_i \\ \rho_e \mathbf{u}_e \\ \mathbb{E}_i \\ \mathbb{E}_e \\ 3\,\mathrm{Sym}(\mathbf{u}_i \mathbb{E}_i) - 2\rho_i \mathbf{u}_i \mathbf{u}_i \mathbf{u}_i + \mathbb{Q}_i \\ 3\,\mathrm{Sym}(\mathbf{u}_e \mathbb{E}_e) - 2\rho_e \mathbf{u}_e \mathbf{u}_e \mathbf{u}_+ \mathbb{Q}_e \end{bmatrix} = \frac{1}{r_g} \begin{bmatrix} 0 \\ 0 \\ \sigma_i(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) \\ \sigma_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) \\ 2\frac{q_i}{m_i}\,\mathrm{Sym}(\rho_i \mathbf{u}_i \mathbf{E} + \mathbb{E}_i \times \mathbf{B}) \\ 2\frac{q_e}{m_e}\,\mathrm{Sym}(\rho_e \mathbf{u}_e \mathbf{E} + \mathbb{E}_e \times \mathbf{B}) \end{bmatrix},$$

$$\partial_t \begin{bmatrix} c\mathbf{B} \\ \mathbf{E} \end{bmatrix} + c\nabla \times \begin{bmatrix} \mathbf{E} \\ -c\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathbf{J}/\varepsilon \end{bmatrix}, \text{ and } \nabla \cdot \begin{bmatrix} c\mathbf{B} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma/\varepsilon \end{bmatrix}.$$

Here Sym denotes the symmetric part of its argument tensor (obtained by averaging over all permutations of subscripts), and $\mathbb{Q}_s := \rho_s \langle \mathbf{c}_s \mathbf{c}_s \mathbf{c}_s \rangle$ is the generalized heat tensor, often assumed to be zero; here $\mathbf{c} := \mathbf{v} - \mathbf{u}$ is the thermal velocity.

We note that the gas-dynamic energy of a species is one half the trace of its energy tensor. The ion pressure tensor $\mathbb{P}_i = \mathbb{E}_i - \rho_i \mathbf{u}_i \mathbf{u}_i$ obeys the evolution equation

$$(d_t^{\mathbf{u}_i})\mathbb{P}_i + (\nabla \cdot \mathbf{u}_i)\mathbb{P}_i + 2\,\mathrm{Sym}(\mathbb{P}_i \cdot \nabla \mathbf{u}_i) + \nabla \cdot \mathbb{Q}_i = 2\,\mathrm{Sym}(\frac{q_i}{m_i}\mathbb{P}_i \times \mathbf{B}),$$

and likewise for the electron fluid. The ion and electron pressure tensors should remain (symmetric) positive definite.

1.5 Boltzmann model.

The Boltzmann equation is an evolution equation for the particle density function which says that particle density is conserved in phase space and is governed by the Lorentz force.

$$\partial_t f_s + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \frac{1}{r_g} \nabla_{\tilde{\mathbf{v}}} \cdot \left(\frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right) = C_s;$$

here s is a species index, $\tilde{\mathbf{v}} = \gamma \mathbf{v} \approx \mathbf{v}$ is (proper) velocity in phase space ($\gamma = \left(1 - (v/c)^2\right)^{-1/2} \approx 1$), $f_s(t, \mathbf{x}, \tilde{\mathbf{v}})$ is particle density as a function of the independent variables, and C_s is a collision operator which is a function of $\{\tilde{\mathbf{v}} \mapsto f_p(t, \mathbf{x}, \tilde{\mathbf{v}})\}_p$, where p ranges over all species. The collisionless Boltzmann equation (alias *Vlasov equation*) asserts that $C_s = 0$.

The Boltzmann equation is coupled to Maxwell's equations:

$$\begin{aligned}
&\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, && \nabla \cdot \mathbf{B} = 0, && \mathbf{J} = \sum_s \int_{\mathbf{v}} f_s q_s \mathbf{v}, \\
&\partial_t \mathbf{E} = c^2 \nabla \times B - \mathbf{J}/\varepsilon, && \nabla \cdot \mathbf{E} = \sigma, && \sigma = \sum_s \int_{\mathbf{v}} f_s q_s.
\end{aligned}$$

1.6 Particle model.

The equations of the particle model are Maxwell's equations and the Lorentz force to govern particle motion:

$$\begin{aligned}
&\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, && \nabla \cdot \mathbf{B} = 0, \\
&\partial_t \mathbf{E} = c^2 \nabla \times B - \mathbf{J}/\varepsilon, && \nabla \cdot \mathbf{E} = \sigma/\varepsilon, \\
&d_t(\gamma \mathbf{v}_p) = \frac{1}{r_g} \frac{q_p}{m_p} \Big(\mathbf{E}(\mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(\mathbf{x}_p) \Big), && d_t \mathbf{x}_p = \mathbf{v}_p, \\
&\mathbf{J} = \sum_p S_p(\mathbf{x}_p) q_p \mathbf{v}_p, && \sigma = \sum_p S_p(\mathbf{x}_p) q_p;
\end{aligned}$$

here p denotes particle index, $\gamma = \left(1 - (v/c)^2\right)^{-1/2} \approx 1$, and S_p denotes the spatial charge distribution of a single particle (e.g. an impulse function). (As above, in the nondimensionalization r_g is the nondimensionalized gyroradius of a typical ion.)

2 GEM magnetic reconnection challenge problem.

2.1 Computational domain.

The computational domain is the rectangular domain $[-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$, where $L_x = 8\pi$ and $L_y = 4\pi$. The problem is symmetric under reflection across either the horizontal or vertical axis.

Remark: in the GEM literature the vertical axis is customarily named z and the out-of-plane axis is named y, opposite to the convention followed here.

2.2 Boundary conditions.

The domain is periodic in the x-axis. The boundaries perpendicular to the y-axis are thermally insulating conducting wall boundaries. A conducting wall boundary is a solid wall boundary (with slip boundary conditions in the case of ideal plasma) for the fluid variables, and the electric field at the boundary has no component parallel to the boundary. Assuming the ideal MHD Ohm's law, this implies that at the conducting boundary the magnetic field must be parallel to the boundary. So at the conducting wall boundaries

$$\begin{aligned}
\partial_y u_{x,s} &= 0, & \partial_y B_x &= 0, & E_x &= 0, & \partial_y \rho_s &= 0, \\
u_{y,s} &= 0, & B_y &= 0, & \partial_y E_y &= 0, & \partial_y p_s &= 0 = \partial_y \tilde{p}, \\
\partial_y u_{z,s} &= 0, & \partial_y B_z &= 0, & E_z &= 0, & \partial_y \mathcal{E}_s &= 0 = \partial_y \tilde{\mathcal{E}}.
\end{aligned}$$

[Insert rules for pressure and energy tensor and divergence cleaning potentials.]

[Insert ghost cell rules.]

[Insert axiomatic derivation of conducting wall boundary conditions (in an appendix?).]

2.3 Model Parameters.

In the GEM problem the model parameters are

$$m_i/m_e = 25, \qquad \mu_0 = 1.$$

In their two-fluid simulations, [Loverich05] and [Hakim06] choose a speed of light

$$c = 10 B_0,$$

i.e. 10 times the Alfvén speed.

For our nondimensionalization parameters we assumed that the gyroradius of the ions, the Debye length, and the light speed are all unity.

2.4 Initial conditions.

The initial conditions are a perturbed Harris sheet equilibrium. The unperturbed equilibrium is given by

$$\mathbf{B}(y) = B_0 \tanh(y/\lambda)\mathbf{e}_x, \qquad p(y) = \frac{B_0^2}{2n_0} n(y),$$
$$n(y) = n_i(y) = n_e(y) = n_0(1/5 + \operatorname{sech}^2(y/\lambda)), \qquad p_e(y) = p(y)/6,$$
$$\mathbf{E} = 0, \qquad p_i(y) = 5p(y)/6.$$

On top of this the magnetic field is perturbed by

$$\delta\mathbf{B} = -\mathbf{e}_z \times \nabla(\psi), \text{ where}$$
$$\psi(x,y) = \psi_0 \cos(2\pi x/L_x)\cos(\pi y/L_y).$$

In the GEM problem the initial condition constants are

$$\lambda = 0.5, \qquad B_0 = 1,$$
$$n_0 = 1, \qquad \psi_0 = B_0/10.$$

[Hakim06] instead sets $B_0 = 0.1$ (so that the speed of light will be unity when it is 10 times the Alfvén speed). His solution maps onto the GEM variables via

$$\mathbf{B} = .1\mathbf{B}_{\mathrm{GEM}}, \qquad t = 10t_{\mathrm{GEM}}, \qquad p = .01p_{\mathrm{GEM}}, \qquad (\rho_s) = (\rho_s)_{\mathrm{GEM}},$$
$$\mathbf{E} = .01\mathbf{E}_{\mathrm{GEM}}, \qquad \mathbf{u} = .1\mathbf{u}_{\mathrm{GEM}}, \qquad \mathcal{E} = .01\mathcal{E}_{\mathrm{GEM}}, \qquad (n_s) = (n_s)_{\mathrm{GEM}},$$

and so forth.

In his two-fluid simulations, [Hakim06] assumes that the initial current is carried only by the electrons and neglects the portion of the current caused by the initial perturbation:

$$\mathbf{J}_e = \nabla \times \mathbf{B} = -\frac{B_0}{\lambda}\operatorname{sech}^2(y/\lambda)\mathbf{e}_z.$$

In the case of a mass ratio near unity and large perturbation, we prefer a more exact calculation of the initial velocities.

We calculate the precise initial current:

$$\begin{aligned}
\mathbf{J} =& \nabla \times (\mathbf{B} + \delta\mathbf{B}). \\
\nabla \times \mathbf{B} =& -\frac{B_0}{\lambda}\operatorname{sech}^2(y/\lambda)\mathbf{e}_z. \\
\delta\mathbf{B} =& -\mathbf{e}_z \times \nabla\psi = \begin{bmatrix} \partial_y\psi \\ -\partial_x\psi \\ 0 \end{bmatrix} = \begin{bmatrix} -\psi_0(\pi/L_y)\cos(2\pi x/L_x)\sin(\pi y/L_y) \\ \psi_0(2\pi/L_x)\sin(2\pi x/L_x)\cos(\pi y/L_y) \\ 0 \end{bmatrix} \\
\nabla \times \delta\mathbf{B} =& -(\nabla^2\psi)\mathbf{e}_z = \psi_0((2\pi/L_x)^2 + (\pi/L_y)^2)\cos(2\pi x/L_x)\cos(\pi y/L_y)\mathbf{e}_z. \\
\mathbf{J} =& (-\frac{B_0}{\lambda}\operatorname{sech}^2(y/\lambda) - \nabla^2\psi)\mathbf{e}_z.
\end{aligned}$$

From the current we calculate the initial species currents and velocities. Assuming zero initial net momentum gives the system

$$\mathbf{J} = \mathbf{J}_i + \mathbf{J}_e,$$
$$0 = m_i \mathbf{J}_i - m_e \mathbf{J}_e,$$

whose solution is

$$\mathbf{J}_i = \frac{m_e}{m_i + m_e}\mathbf{J},$$
$$\mathbf{J}_e = \frac{m_i}{m_i + m_e}\mathbf{J}.$$

So the initial momentum of each species is:

$$\rho_i \mathbf{u}_i = m_i \mathbf{J}_i / e = m\mathbf{J}/e,$$
$$\rho_e \mathbf{u}_e = -m_e \mathbf{J}_i / e = -m\mathbf{J}/e,$$

where $m = \frac{m_i m_e}{m_i + m_e}$ is the *reduced mass*.

For the ten-moment model we assumed the same values as the five-moment model, taking the pressure tensor to be initially isotropic.

3 X-point

Analysis of the X-point (the origin) is the key to understanding the GEM problem.

3.1 X-point electric field gives rate of reconnection.

The reconnected flux F_{recon} is typically defined to be the loss of magnetic flux through the positive y-axis:

$$F_{\text{left}}(t) := \int_0^{y_{\max}} B_1 \, dy, \qquad F_{\text{recon}}(t) := F_{\text{left}}(0) - F_{\text{left}}(t).$$

I claim that the rate of reconnection is minus the value of the out-of-plane component of the electric field at the X-point. (This confirms the theoretical fact that an MHD model which only includes the $\mathbf{B} \times \mathbf{u}$ and Hall terms in Ohm's law cannot give fast reconnection, since by symmetries both these terms must vanish at the origin.) Indeed, using Faraday's law, $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$,

$$d_t F_{\text{recon}}(t) = -d_t F_{\text{left}}(t) = -\int_0^{y_{\max}} \partial_t B_1 \, dy = \int_0^{y_{\max}} \partial_y E_3 \, dy = -E_3(0),$$

since E_3 is zero at the conducting wall. *Remark: if we instead denote the out-of-plane axis as the y axis, then*

$$d_t F_{\text{recon}}(t) = E_y(0),$$

i.e., without a minus sign.

3.2 Symmetries.

The GEM problem is invariant under reflection across the horizontal and vertical axes. Note that the magnetic field **B** is a *pseudovector*, which means that it is negated under a reflection (i.e. an orientation-reversing isometry). As a result, all tensor and pseudotensor components must be even or odd when reflected across any pair of axes.

Sometimes the GEM problem is modified by adding to the magnetic field a *guide field*, i.e. a constant background magnetic field in the out-of-plane direction. In this case the GEM problem is no longer symmetric under horizontal and vertical reflections, but it is still symmetric under 180-degree rotations.

As a result, all tensor and pseudotensor components must still be even or odd functions when restricted to any line through the origin. Thus, components which are odd along some line must therefore be zero at the origin, and components which are even are constant at the origin up to first order. Taking tensor products of vectors indicates these symmetries for components of tensors of arbitrary order.

We remark that the GEM problem is very sensitive to initial conditions and in particular is subject to symmetry-breaking. Neglecting to enforce symmetry in simulations can admit the manifestation of secondary instabilities that would otherwise not be seen.

We now assume (enforced) symmetry and study the evolution of quantities near the origin. Choose a species, say the ions. Write its momentum equation, solved for **E**:

$$\mathbf{E} = -\frac{\mathbf{R}_i}{en_i} + \mathbf{u}_i \times \mathbf{B} + \frac{\nabla \cdot P_i}{en_i} + \frac{m_i}{e} d_t \mathbf{u}_i;$$

here d_t denotes the material derivative with respect to the flow $\mathbf{v}_i$, and $\mathbf{R}_i$ denotes resistive drag force, typically proportional to (or at least a linear function of) the current.

Evaluate the mometum equation at the origin. Even in the presence of a guide field, vectors and pseudovectors must be out of the plane. So the cross product vanishes, $\mathbf{v}_i.\nabla \mathbf{v}_i = 0$, and the momentum equation becomes

$$E_3 = -\frac{R_{i3}}{en_i} + \frac{\partial_x P_{i13} + \partial_y P_{i23}}{en_i} + \frac{m_i}{e} \partial_t u_{i3}.$$

(This is a single-species form of Ohm's law.) In the collisionless model the resistive term vanishes. If the pressure tensor is isotropic its divergence must vanish. (And it will vanish at the origin even if the pressure tensor is merely *laterally* isotropic, i.e. invariant under rotations around the magnetic field vector.) So, in a collisionless, isotropic model of plasma, the velocity at the origin must exactly track the reconnected flux.

3.3 Symmetric pair plasma.

If we choose equal temperatures and equal masses for the two species, then (in the absence of a guide field) we have complete symmetry between species. In this case the GEM problem is invariant under exchange of species coupled with reflection along the out-of-plane axis. The magnetic field remains in the plane, and the electric field remains everywhere perpendicular to the plane. So we can simulate the system with half the number of variables. (For this symmetric pair plasma case, typically $\mathbf{R}_i = -\alpha(en_i)\mathbf{u}_i$.) In the (laterally) isotropic model Ohm's law at the origin becomes

$$E_3 = \alpha u_{i3} + \frac{m_i}{e}\partial_t u_{i3}.$$

Plugging this into Maxwell's evolution equation for electric field gives a damped harmonic oscillator equation for u_{i3} forced by $\nabla \times \mathbf{B}$.

Integrating over time relates the reconnected flux to the species velocity at the origin

$$\mathrm{Flux}_{\mathrm{reconnected}}(t) = \frac{m_i}{e}u_{i3}(0) - u_{i3}(t) - \alpha\int_0^t u_{i3}.$$

A Nondimensionalization.

Physical constants that define a plasma are:

1. e, the magnitude of the charge of an electron,
2. m_i, m_e, $m := (m_i^{-1} + m_e^{-1})^{-1}$, the ion, electron, and reduced mass, and
3. c, the speed of light.

Three fundamental parameters that characterize the state of a plasma are:

1. n_0, a typical particle density,
2. T_0, a typical temperature (often per species), and
3. B_0, a typical magnetic field strength.

Subsidiary parameters derived from these are the gyrofrequencies $\omega_{g,s}$, the plasma frequencies $\omega_{p,s}$, the thermal velocities $v_{t,s}$, the Alfvén velocity v_A, the gyroradii $r_{g,s}$, the Debye length λ_D, and the inertial lengths (i.e. skin depths) δ_s:

1. $\omega_{g,s} = \frac{eB_0}{m_s}$,

2. $\omega_{p,s}^2 = \frac{n_0 e^2}{\varepsilon_0 m_s}$,

3. $v_{t,s}^2 = \frac{T_0}{m_s}$,

4. $v_A^2 = \frac{B_0^2}{\mu_0 m_s n_0}$,

5. $r_{g,s} = \frac{v_{t,s}}{\omega_{g,s}} = \frac{m_s v_{t,s}}{e B_0}$,

6. $\lambda_D^2 = \left(\frac{v_{t,s}}{\omega_{p,s}}\right)^2 = \frac{\varepsilon_0 T_0}{n_0 e^2}$,

7. $\delta_s^2 = \left(\frac{c}{\omega_{p,s}}\right)^2 = \left(\frac{v_A}{\omega_{g,s}}\right)^2 = \frac{m_s}{\mu_0 n_0 e^2}$

These parameters pop out of a generic nondimensionalization of the particle (or Boltzmann or 2-fluid) equations:

$$\begin{aligned}
&\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, && \nabla \cdot \mathbf{B} = 0,\\
&\partial_t \mathbf{E} = c^2 \nabla \times B - \mathbf{J}/\varepsilon, && \nabla \cdot \mathbf{E} = \sigma/\varepsilon,\\
&d_t(\gamma \mathbf{v}_p) = (t_0 \omega_g) \frac{q_p}{m_p} \Big(\mathbf{E}(\mathbf{x}_p) + \mathbf{v}_p \times \mathbf{B}(\mathbf{x}_p)\Big), && d_t \mathbf{x}_p = \mathbf{v}_p,\\
&\mathbf{J} = \sum_p S_p(\mathbf{x}_p) q_p \mathbf{v}_p, && \sigma = \sum_p S_p(\mathbf{x}_p) q_p;
\end{aligned}$$

here $t_0 \omega_g = t_0 \frac{q_0 B_0}{m_0}$ is the gyrofrequency nondimensionalized by a choice of t_0 (which can be chosen to be the gyroperiod in order to set this factor to unity) and $\frac{1}{\varepsilon} = \frac{x_0 n_0 e}{v_0 B_0 \varepsilon_0} = t_0 \frac{e B_0}{m} \frac{\mu_0 m_0 n_0}{B_0^2} c^2 = (t_0 \omega_g) \left(\frac{c}{v_A}\right)^2$. Note that we can also write $(t_0 \omega_g) = \frac{x_0}{r_g}$ and $\frac{c}{v_A} = \frac{r_g}{\lambda_D} = \frac{\omega_p}{\omega_g}$.

A.1 MHD

Ideal MHD takes $c \to \infty$ and $r_g \to 0$ and therefore has only one free parameter (μ_0) rather than three. To nondimensionalize the MHD equations we make the substitution

$$\mathbf{B} = \sqrt{\mu_0} \hat{\mathbf{B}};$$

this eliminates μ_0. More generically, we can nondimensionalize using a typical Alfvén speed, density, and time scale:

$$\begin{aligned}
&v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}, && \mathbf{B} = B_0 \hat{\mathbf{B}}, && \rho = \rho_0 \hat{\rho}, && t = t_0 \hat{t},\\
&\mathbf{u} = v_A \hat{\mathbf{u}}, && p = \rho_0 v_A^2 \hat{p}, && \mathcal{E} = \rho_0 v_A^2 \hat{\mathcal{E}}, && x = v_A t_0 \hat{x}.
\end{aligned}$$

B Conducting wall boundaries.

References

[Birn01] Birn, J., et al., *Geospace Environmental Modeling (GEM) magnetic reconnection challenge*, J. Geophys. Res., 106(A3), 3715–3720, 2001.

[Hakim06] A. Hakim, J. Loverich, U. Shumlak, *A high resolution wave propagation scheme for ideal two-fluid plasma equations*, Journal of Computational Physics, 219 (2006) 418-442.

[Hakim08] A. Hakim, *Extended MHD modelling with the ten-moment equations,* Journal of Fusion Energy, 27:36-43, 2008.

[Loverich05] J. Loverich, A. Hakim, U. Shumlak, *A discontinuous Galerkin method for ideal two-fluid plasma equations,* Preprint submitted to Journal of Computational Physics, posted at `http://www.john-loverich.com/Loverich-JCP-2005.pdf`, 13 December 2005.