Determinants

$$\det(a_{..}) = \sum_{\sigma\in Sym(\mathbb{N}_n)} (-1)^\sigma \prod_{i\in\mathbb{N}_n} a_{i\sigma(i)}$$

where:

$\mathbb{N}_n = \{1..n\}$

$Sym(\mathbb{N}_n)$ = set of "symmetries" of $\mathbb{N}_n$
= set of permutations of $\mathbb{N}_n$.

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

Expansion by cofactors:

$$\det(a_{..}) = \sum_{j=1}^{n} a_{kj} \boxed{\sum_{\substack{\sigma\in Sym(\mathbb{N}_n) \\ \sigma(k)=j}} (-1)^\sigma \prod_{\substack{i\in\mathbb{N}_n \\ i\neq k}} a_{i\sigma(i)}}$$

(Call A^{jk})

Let $\sigma\in Sym(\mathbb{N}_n)$ s.t. $\sigma(k)=j$.

Let $\tau\in Sym(\mathbb{N}_{n-1})$ be defined by:

$$\tau = \underbrace{(k\to(k+1)\to\cdots\to n)}_{\text{Call } C_k^{-1}} \to \sigma \to \underbrace{(n\to(n-1)\to\cdots\to j)}_{\text{Call } C_j}$$

Let $\rho = C_k^{-1} C_j$

Write $a_{i\sigma(i)} = b_{C_k(i)\,C_j(\sigma(i))} = b_{C_k(i)\,\tau(C_k(i))}$

So $b_{l\tau(l)} = a_{C_k^{-1}(l)\,\sigma(C_k^{-1}(l))}$

Observe $(-1)^\tau = (-1)^{C_k^{-1}}\cdot(-1)^\sigma\cdot(-1)^{C_j}$

$= (-1)^{(n-k)}\cdot(-1)^\sigma\cdot(-1)^{(n-j)}$

$= (-1)^{k+j}\cdot(-1)^\sigma$

Thus:

$$A^{jk} = (-1)^{k+j}\sum_{\tau\in Sym(\mathbb{N}_{n-1})} (-1)^\tau \cdot \prod_{\substack{i\in\mathbb{N}_n \\ i\neq k}} b_{C_k(i)\,\tau(C_k(i))}$$

$$= \underbrace{(-1)^{k+j}\sum_{\tau\in Sym(\mathbb{N}_{n-1})} (-1)^\tau \prod_{i\in\mathbb{N}_{n-1}} b_{i\tau(i)}}_{\substack{\text{Determinant of matrix} \\ \text{obtained by deleting} \\ \text{the } k^{th} \text{ row and } j^{th} \\ \text{column from } (a_{..}).}}$$

A^{jk} is the transpose of the matrix of cofactors.

So $\boxed{\det(a_{..}) = \sum_{j=1}^{n} a_{kj}A^{jk}}$.

Properties of determinant

1. Linear in each row (or column)
2. Swapping two rows negates value.
3. If two rows (or columns) are identical, the value is zero.
4. If one row is a multiple of another row the value is zero.
5. Adding a multiple of one row to another row does not change the determinant.
6. Transposing the matrix does not change its value.

Pf 1, 2, and 6 are obvious from def. To prove 3-5 it suffices to prove the follow facts for bilinear forms:

Thm Let $(\cdot,\cdot)$ be a bilinear form. Then statements 2-5 above are equivalent.

Pf (universal quantifiers everywhere)

(2) $(a,b)+(b,a)=0$

$\Rightarrow (a,a)+(a,a)=0$

(3) $\Rightarrow (a,a)=0$

(4) $\Rightarrow (a,sa)=0$ (s is a scalar)

$\Rightarrow (a,b) = (a,b)+(a,sa)$

(5) $\Rightarrow (a,b) = (a,b+sa)$

Clearly (5) $\Rightarrow$ (4) $\Rightarrow$ (3).

(3 $\Rightarrow$ 2): $(a,b) = (a,b)-(a,a)$

$= (a,b-a)+(b-a,b-a)$

$= (b,b-a)-(b,b)$

$= (b,-a)$

$= -(b,a)$. ∎

7. The determinant of the identity matrix is 1. (Clear.)

Thm A proposed determinant function $d(a_{..})$ satisfying (1), (2-5), and (7) is unique.

Pf $d(a_{..})$ may be evaluated by row reducing the matrix $(a_{..})$.

8. The determinant of a product is the product of the determinants.

Pf Can verify that multiplying a matrix A on the left by an elementary matrix E multiplies the determinant of A by the determinant of E, using the rules above.

Motivated development of the determinant function

Want to determine two things about an ordered list of n vectors in $\mathbb{R}^n$:

① The volume spanned.
(The volume of the parallelpiped consisting of linear combinations of the n vectors with coefficients in the interval $[0,1]$.)

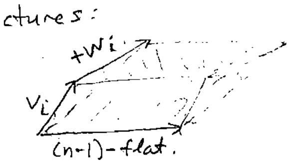
② The orientation of the parallelpiped.
A set of n vectors is <u>positively oriented</u> if these vectors may be changed in a continuous fashion, without ever allowing the volume to go to zero, so that the vectors end up being the standard basis.
If their volume is zero, their orientation is 0 or undefined
Otherwise they are negatively oriented.
The sign of the orientation of the n vectors is defined to be 1, 0, or -1 respectively.
A formal definition of the determinant will allow us to rigorously justify the concept of orientation – i.e. to show that there really is something called orientation with the desired properties.

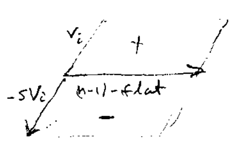
We "define" the determinant of an ordered list $V_1 \dots V_n$ of column vectors in $\mathbb{R}^n$ to be the volume they span times the sign of their orientation.

I claim that it is fairly easy to see that properties (1), (3), and (7) of a determinant should be true;

(1) Linear in each column.

Pictures:





Linear in each row:
Not obvious to me.
But easy to see that the determinant scales in each row. Scaling of a row is stretching one of the coordinate axes.

(3) If two column vectors of V are identical, the volume is zero since the vectors do not span an n-dim'l space

If two rows of V are identical it is also fairly easy to see that the volume must be zero.
Let the j^{th} & k^{th} rows be identical.
This implies that every vector in V is in the $(n-1)$-dim'l subspace

$$\mathbb{R}\underline{e}_1 + \dots + \mathbb{R}\underline{e}_{j-1} + \underline{e}_j + \mathbb{R}\underline{e}_{j+1} + \dots + \mathbb{R}\underline{e}_{k-1} + \underline{e}_k + \dots + \mathbb{R}\underline{e}$$

So the volume is zero.

Since (1) and (3) are true both for columns and for rows, and since (1), (3), and (7) are sufficient to justify evaluation of the determinant by row/column reduction,
(8) It follows that the determinant should be invariant under matrix transpose.

Deduction of formula from properties

Since $\det(V)$ is linear in k^{th} row, write:

$$\det(V) = \sum_i V_{ki} \cdot f_{ki}(V')$$

where V' is the remaining columns of V and f_{ki} is some expression.

Since this can be done for any row k, it follows that the determinant must be a linear combination of products of elements from V. Each product contains ~~at most~~ one element from each row. (In fact, since swapping rows merely negates the determinant, it follows by symmetry principles that each product contains exactly one element from each row.) don't need

At this point just apply the determinant to a matrix with ones in locations referenced by the product. Everything follows.

Since the determinant must scale in each row, each product can contain at most one element from each column.
So each product must contain exactly one element from each row & each column.

It remains to establish the coefficient of each product.
Consider an arbitrary term of the determinant. The matrix entries it references are the nonzero entries of a unique permutation matrix.
By property (2), the determinant of a permutation matrix must be 1 for an even permutation and -1 for an odd permutation, i.e. the sign of the permutation. So the coefficient of each term must be the sign of the corresponding permutation.

Deriving the determinant

1. Define intuitively/geometrically
2. Argue for
   - multilinearity
   - alternating character
   - value on identity
3. Deduce & argue for other properties
   - equivalent criteria for alternating
   - how to evaluate (hence uniqueness)
   - value for elementary matrices
   - homomorphic property
   - value for permutation matrices and matrices with a single 1 in each row. (Need permutation parity).
4. Deduce formula for determinant using:
   - multilinearity to argue for sum of products
   - value for matrices with a single 1 in each row.
5. Prove that this formula has the desired properties, and hence the unique determinant actually exists.
6. Derive expansion by cofactors.
   - deduce existence of inverse
   - deduce Cramer's rule
   - develop cross product as application.
7. Use the notion of continuous homomorphism to justify a rigorous notion of orientation.