Polynomial Interpolation

* Find the n^{th} order polynomial $p(x)$ satisfying:

$$p(x_i) = f(x_i) \quad i = 0..n.$$

- **Lagrange formula:**

Let $\tilde{l}_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} (x - x_i)$

Let $l_k(x) = \dfrac{\tilde{l}_k(x)}{\tilde{l}_k(x_k)} = \prod_{\substack{i=0 \\ i\neq k}}^{n} \left(\dfrac{x-x_i}{x_k - x_i}\right)$

Then $p(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x)$

i.e. $$\boxed{p(x) = \sum_{k=0}^{n} f(x_k) \prod_{\substack{i=0 \\ i\neq k}}^{n} \left(\frac{x-x_i}{x_k - x_i}\right)}$$

- **Newton formula:** (Recursive)

Let $p_k(x)$ be the k^{th} order polynomial satisfying $p_k(x_i) = f(x_i)$, $i = 0..k$.

Let $\omega_k(x) = \prod_{i=0}^{k-1} (x - x_i)$

Observe that for some A_k

$$P_k(x) = P_{k-1}(x) + A_k \omega_k(x)$$

More verbosely:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} A_k \omega_k(x) \\ &= \sum_{k=0}^{n} A_k \prod_{i=0}^{k-1} (x - x_i) \\ &= A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) \\ &\quad + \dots + A_n(x-x_0)\cdots(x-x_{n-1}) \end{aligned}$$

Observe that A_k is the coefficient of the leading order term of $P_k(x)$.

Recall that the interpolating polynomial $P_k(x)$ is determined by the value of the function $f(x)$ at the points $x_0 \dots x_k$. (and therefore so is its leading coefficient A_k), independent of the order in which the points $x_0, \dots, x_k$ are listed.

Therefore A_k is a function of the values of f at $x_0, \dots, x_k$ - a function whose value is invariant under permutations of its arguments.

Hence we define:

$f[x_0, \dots, x_k] := A_k$, the k^{th} divided difference of $f(x)$ at the points $x_0, \dots, x_k$.

and we note that $f[x_0, \dots, x_k]$ is invariant under permutation of its arguments.

(Newton formula cont.)

Thus we have:

$$\boxed{P_n(x) = \sum_{k=0}^{n} f[x_0, \dots, x_k] \prod_{i=0}^{k-1} (x - x_i)}$$

Observe that:

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f(x_2) - [f[x_0] + f[x_0, x_1](x_2 - x_0)]}{(x_2 - x_0)(x_2 - x_1)} \\ &= \frac{f[x_0, x_2] - f[x_0, x_1]}{(x_2 - x_1)} \\ &\left(= \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}\right) \end{aligned}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_0, x_1, x_3] - f[x_0, x_1, x_2]}{x_3 - x_2} \quad \text{(exercise)}$$

Claim:

$$\boxed{f[x_0 \cdots x_k] = \frac{f[x_1 \cdots x_k] - f[x_0 \cdots x_{k-1}]}{x_k - x_0}}$$

Pf

Note $P_{k-1}(x)$ is the $(k-1)^{th}$ order polynomial satisfying $P_{k-1}(x_i) = f(x_i)$ $\quad i = 0..k-1$

Let $q_{k-1}(x)$ be the $(k-1)^{th}$ order polynomial satisfying $q_{k-1}(x_i) = f(x_i)$ $\quad i = 1..k.$

Observe that

$$\begin{aligned} P_k(x) &= P_{k-1}(x) + \frac{(x - x_0)}{(x_k - x_0)} [q_{k-1}(x) - P_{k-1}(x)] \\ &= \frac{(x-x_0)\, q_{k-1}(x) - (x - x_k)\, P_{k-1}(x)}{(x_k - x_0)} \end{aligned}$$

Taking the leading coefficients of each side yields the claim.

Thm Error of interpolation (1)

Let $x, x_0, \cdots, x_n \in [a,b]$ and $f \in C^{n+1}[a,b]$

Then $\exists\, \xi$ s.t. $\min\{x, x_0, \cdots, x_n\} < \xi < \max\{x, x_0, \cdots, x_n\}$

s.t. $$\boxed{f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega(x)}$$

where $\omega(x) = \prod_{k=0}^{n} (x - x_k)$

and $p_n(x)$ is the interpolating polynomial.

Pf WLOG $x \notin \{x_1, \cdots, x_n\}$

WLOG $p_n(t) = 0$ $(\forall t)$

(To see this, let $\tilde{f}(t) = f(t) - p_n(t)$.
So $\tilde{f}^{(n+1)}(t) = f^{(n+1)}(t)$.)

We want to express $f(x)$ in terms of $f^{(n+1)}$. f has $n+1$ zeros.

If we had a function with $n+2$ zeros in $[a,b]$, Rolle's theorem would guarantee that its $(n+1)^{th}$ derivative has a zero in $[a,b]$.

Let $$g(t) = f(t) - \frac{f(x)}{\omega(x)}\,\omega(t)$$

Observe that g is zero $\forall t \in \{x, x_0, \cdots, x_n\}$

So $\exists\, \xi$ s.t. $g^{(n+1)}(\xi) = 0$

Note $\omega^{(n+1)}(\xi) = (n+1)!$

So $$f^{(n+1)}(\xi) = \frac{f(x)}{\omega(x)}(n+1)!$$

i.e. $$f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega(x).$$

Thm Error of Interpolation (2)

$$\boxed{f(x) - p_n(x) = f[x_0, \cdots, x_n, x]\,\omega(x)}$$

where $\omega(x) = \prod_{k=0}^{n} (x - x_k)$

Pf Let $x_{n+1} = x$.

Let p_{n+1} interpolate $\{x_0, \cdots, x_{n+1}\}$.

Have $p_{n+1}(t) = p_n(t) + f[x_0, \cdots, x_n] \prod_{k=0}^{n} (t - x_k)$.

But $p_{n+1}(x) = f(x)$.

Thm Relationship of divided differences and derivatives

$$f[x_0, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

Pf By prev. two thms,

$$f[x_0, \cdots, x_{n+1}] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$