Traffic Flow & Burgers' Equation

Let $\rho(x,t)$ = car density = $\frac{\#\text{cars}}{\text{unit length}}$

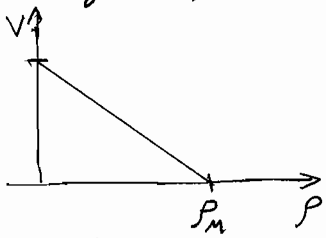
Let $q(x,t)$ = car flux = $\#$cars / unit time

Assume

- $\int \rho\, dx$ is conserved. (Cars conserved.)
- $q = \rho V \leftarrow$ car velocity at x, t
- $\boxed{V = V(\rho)}$

simplest:

$$\frac{V}{V_M} = 1 - \frac{\rho}{\rho_M}$$



(quadratic is more realistic.)

Thus $q = q(\rho) = \rho V(\rho)$

Conservation of cars plus assumption of differentiability gives:

$$\boxed{\rho_t + q_x = 0}$$

But $q_x = \frac{dq}{d\rho}\rho_x$.

Let $\boxed{c(\rho) \equiv \frac{dq}{d\rho}}$ ($c \equiv$ "celerity" of density)

So $\boxed{\rho_t + c(\rho)\rho_x = 0}$

Method of characteristics

$$\left.\begin{aligned}\frac{dX}{dt} &= c(\rho)\\ \frac{d\rho}{dt} &= 0\end{aligned}\right\} \begin{aligned}&X = X_0 + c(\rho_0)t\\ &\rho(X,t) = \rho(X_0,0) = \rho_0(X_0)\end{aligned}$$

Implicit equation for ρ:

$$\boxed{\rho(x,t) = \rho_0(x - c(\rho)t)}$$

Observe $c(\rho) = \frac{dq}{d\rho} = V + \rho\frac{dV}{d\rho}$

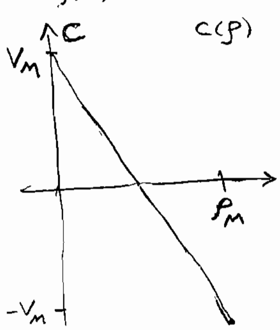
Expect $c(\rho)$ to decrease monotonically.

In the linear case:

$$V = V_M\left(1 - \frac{\rho}{\rho_M}\right), \quad \frac{dV}{d\rho} = -\frac{V_M}{\rho_M}$$

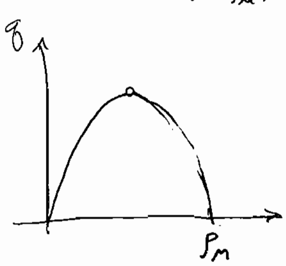
$$\text{So } c(\rho) = V_M\left(1 - \frac{\rho}{\rho_M}\right) + \rho\left(\frac{-V_M}{\rho_M}\right)$$

$$= V_M - 2\frac{V_M}{\rho_M}\rho$$



Also: $q(\rho) = \rho V$.

$$= \rho\left(1 - \frac{\rho}{\rho_M}\right)V_M$$



Discontinuous Solutions (shocks)

Assume conservation of cars for a (moving) control volume.

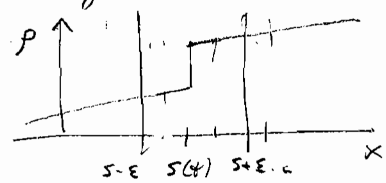
$\rho_t + q_x = 0$

$\int_{x_1}^{x_2} \rho_t\, dx + [q]_{x_1}^{x_2} = 0$

But $\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho\, dx = \int_{x_1}^{x_2} \rho_t\, dx + [\rho\dot{x}]_{x_1(t)}^{x_2(t)}$

$\underbrace{\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho\, dx}_{\text{set} = 0} - \underbrace{[\rho\dot{x}]_{x_1(t)}^{x_2(t)}}_{\dot{x} = \dot{s}} + [q]_{x_1(t)}^{x_2(t)} =$

So $[\rho]\dot{s} = [q]$.



$s(t) \equiv$ shock position.

Assume $\lim_{\varepsilon \to 0} \frac{d}{dt}\int_{s-\varepsilon}^{s+\varepsilon} \rho\, dx = 0.$

OR

$[q - \rho\dot{s}] = 0$

$\Rightarrow [q] = [\rho]\dot{s}$

$\lim_{\varepsilon \to 0} [q - \rho\dot{s}]_{-\varepsilon}^{+\varepsilon} = 0$

(OR, since $q = \rho V(\rho)$, $[\rho(V - \dot{s})] = 0$)

$$q_+ - q_- = (\rho_+ - \rho_-)\dot{s}$$

$$\boxed{\dot{s} = \frac{[q]}{[\rho]} = \frac{q_+ - q_-}{\rho_+ - \rho_-}}$$

Nonviscous Burgers' Equation:

Have $\rho_t + c(\rho)\rho_x = 0$.

Multiply through by $\frac{dc}{d\rho}$.

Get $\frac{dc}{d\rho}\rho_t + c(\rho)\frac{dc}{d\rho}\rho_x = 0$

i.e. $\boxed{c_t + c \cdot c_x = 0}$

Condition for $c(\rho)$ to decrease monotonically

Want $c'(\rho) < 0$.

Recall $q(\rho) = \rho \cdot V(\rho)$

So $c(\rho) \equiv q'(\rho) = V + \rho\frac{dV}{d\rho}$

$$c'(\rho) = q''(\rho) = 2\frac{dV}{d\rho} + \rho\frac{d^2V}{d\rho^2}$$

Need $\boxed{\frac{d^2V}{d\rho^2} < -\frac{2}{\rho}\frac{dV}{d\rho}}$

(True if V is decreasing and is linear or concave down.)

1^{st} order nonlinear PDE (single equation)

2D Given $F(x,y,u_x,u_y,u)=0$

Solve for $u(x,y)$.

Solution

Define:

$$\begin{cases} p \equiv u_x \\ q \equiv u_y \end{cases} \text{ So } p_y = q_x$$

$$G(x,y) \equiv F(x,y,p,q,u)$$

So $G(x,y)=0$.

So $$\frac{\partial G}{\partial x} = 0 = F_x + F_p p_x + F_q \overset{p_y}{q_x} + F_u p$$

$$\frac{\partial G}{\partial y} = 0 = F_y + F_p \overset{q_x}{p_y} + F_q q_y + F_u q$$

Rewriting this system:

$$F_p p_x + F_q p_y = -F_x - F_u p$$

$$F_p q_x + F_q q_y = -F_y - F_u q$$

Defining:

$$\alpha \equiv F_p \qquad \gamma \equiv -F_x - F_u p$$

$$\beta \equiv F_q \qquad \delta \equiv -F_y - F_u q$$

(Note $\alpha, \beta, \gamma, \delta$ are functions of x, y, p, q, u.)

Have:

$$\left.\begin{cases} \alpha p_x + \beta p_y = \gamma \\ \alpha q_x + \beta q_y = \delta \\ p_x - q_y = 0 \end{cases}\right\} \text{3 equations in 3 unknowns } (p, q, \& u)$$

These are quasilinear PDEs for p & q.

Method of characteristics:

$$\left.\begin{cases} \frac{dX}{dt} = \alpha(X,Y,p,q,u) \\ \frac{dY}{dt} = \beta(X,Y,p,q,u) \\ \frac{dp}{dt} = \gamma(\ldots) \\ \frac{dq}{dt} = \delta(\ldots) \\ \frac{dU}{dt} = p\alpha + q\beta \end{cases}\right\} \text{System of 5 nonlinear ODEs.}$$

(Generalization)

N-D

Given $H(\underline{x}, \underline{p}, u) = 0$ "Hamiltonian"

$\underline{x} \equiv (x_1, \ldots, x_n)$

$\underline{p} \equiv (p_1, \ldots, p_n)$, $p_i \equiv \frac{\partial u}{\partial x_i} \equiv u_{,i}$

Find $u(\underline{x})$.

Solution

Observe: $\frac{\partial p_i}{\partial x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial p_j}{\partial x_i} = u_{,ij}$

Let $G(\underline{x}) = H(\underline{x}, \underline{p}, u)$.

Have $G(\underline{x}) = 0$.

So $\frac{\partial}{\partial x_i} G = 0 = H_{x_i} + H_{p_j} \frac{\partial p_j}{\partial x_i} + H_u p_i$

$$\boxed{0 = H_{x_i} + H_{p_j} \frac{\partial p_i}{\partial x_j} + H_u p_i}$$

$\forall i \in \{1, \ldots, n\}$ Quasilinear PDEs for p_i.

Method of characteristics!

$$\frac{dX_j}{dt} = H_{p_j} \quad (j = 1..n)$$

$$\frac{dp_i}{dt} = -H_{x_i} - H_u p_i$$

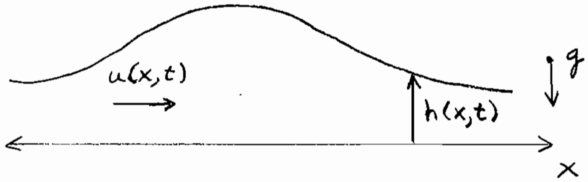
$$\frac{dU}{dt} = p_j H_{p_j}$$

To turn this back into a PDE, note that

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial x_j} \frac{dX_j}{dt} = \frac{\partial p_i}{\partial x_j} H_{p_j}$$

So $\frac{\partial p_i}{\partial x_j} H_{p_j} = -H_{x_i} - H_u p_i$, as wanted.

Shallow water summary



$u(x,t) \equiv$ horizontal velocity.
$h(x,t) \equiv$ height
$g \equiv$ acceleration of gravity.
($\rho \equiv$ density of fluid)

Assumptions

- No viscosity, slip boundary conditions on bottom
- Flat bottom
- * $h \ll \lambda$, depth much smaller than wavelengths
- atmospheric pressure is constant,
- u is independent of y initially

Deductions

- vertical accelerations are negligible. (can additional assumption justified a posteriori)
- u remains independent of y (stable?)
- Shock waves can develop. (This situation contradicts the assumptions that $h \ll \lambda$ and that vertical accelerations are negligible, so it would be nice if we could weaken our assumptions somehow. Perhaps we could show that the assumptions and results continue to hold as averages over large length scales. Perhaps averages weighted by a Gaussian distribution over x with standard deviation much greater than the height of the water.)

Equations (SWE)

Conservation form

$$h_t + (hu)_x = 0 \qquad \text{(mass)}$$
$$(hu)_t + (hu^2 + gh^2/2)_x = 0 \qquad \text{(momentum)}$$

Quasi-Linearized

$$h_t + uh_x + hu_x = 0 \qquad \text{(mass)}$$
$$u_t + uu_x + gh_x = 0 \qquad \text{(momentum)}$$

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{bmatrix} u & h \\ g & u \end{bmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = \underline{0}$$

$$\underline{U}_t + \underline{A} \cdot \underline{U}_x = \underline{0}$$

Linearized : $u = \varepsilon \tilde{u}$, $h = h_0 + \varepsilon \tilde{h}$

$$\begin{pmatrix} \tilde{h} \\ \tilde{u} \end{pmatrix}_t + \begin{bmatrix} 0 & h_0 \\ g & 0 \end{bmatrix} \begin{pmatrix} \tilde{h} \\ \tilde{u} \end{pmatrix}_x = O(\varepsilon)$$

$$\Rightarrow \tilde{h}_t = -h_0 \tilde{u}_x, \quad \tilde{u}_t = -g\tilde{h}_x$$
$$\Rightarrow \tilde{h}_{tt} = h_0 g \tilde{h}_{xx} = c^2 \tilde{h}_{xx}$$

where $c = \sqrt{h_0 g}$ = speed of waves.

Characteristics

eigenvalues of A:
$$\lambda_\pm = u \pm \sqrt{gh} = u \pm c$$
eigenvectors of A (left):
$$\ell_+^T = (1, \tfrac{c}{g})$$
$$\ell_-^T = (1, -\tfrac{c}{g})$$

Characteristic equations

$$\frac{dX_+}{dt} = u + c \qquad \frac{d^+}{dt}(u + 2c) = 0$$
$$\frac{dX_-}{dt} = u - c \qquad \frac{d^-}{dt}(u - 2c) = 0$$

Shock conditions

$$[uh] = \dot{s}[h] \quad \Leftrightarrow \quad [Uh] = 0$$
$$[u^2h + gh^2/2] = \dot{s}[uh] \Leftrightarrow [U^2h + gh^2/2] = 0$$

where $U = u - \dot{s}$

Derivation of Riemann invariants

Recall: $\begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{bmatrix} u & h \\ g & u \end{bmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$

eigenvalues

$$\det \begin{pmatrix} u-\lambda & h \\ g & u-\lambda \end{pmatrix} = 0$$

$$(u-\lambda)^2 = gh.$$

$$\lambda_\pm = u \pm \sqrt{gh} = u \pm c, \text{ where } c = \sqrt{gh}$$

eigenvectors

Want $\ell^T (A - I\lambda) = \underline{0}$

So $\ell_\pm^T \begin{bmatrix} \mp c & h \\ g & \mp c \end{bmatrix} = \underline{0}$

Let $\ell_+^T = \left(1, \frac{c}{g}\right)$

$\ell_-^T = \left(1, \frac{-c}{g}\right)$

Have:

$$\underline{\ell}_\pm^T \left[\underline{U}_t + \lambda_\pm \underline{U}_x \right] = 0$$

characteristic equations

$$\frac{dX_+}{dt} = \lambda_+ = u + c \qquad 0 = \underline{\ell}_+^T \frac{d^+ \underline{U}}{dt} = \frac{d^+}{dt} h + \frac{c}{g} \frac{d^+}{dt} u$$

$$\frac{dX_-}{dt} = \lambda_- = u - c \qquad 0 = \underline{\ell}_-^T \frac{d^- \underline{U}}{dt} = \frac{d^- h}{dt} - \frac{c}{g} \frac{d^-}{dt} u$$

Riemann invariants

$$0 = \frac{d}{dt} h \pm \frac{c}{g} \frac{d}{dt} u$$

$$= \frac{d}{dt}\left(\frac{c^2}{g}\right) \pm \frac{c}{g} \frac{d}{dt} u$$

$$0 = 2c \frac{dc}{dt} \pm c \frac{du}{dt}$$

$$0 = \frac{d}{dt}(2c \pm u)$$

Characteristics w/ Riemann invariants

$$\frac{dX_+}{dt} = u + c \qquad \frac{d^+}{dt}(u + 2c) = 0$$

$$\frac{dX_-}{dt} = u - c \qquad \frac{d^-}{dt}(u - 2c) = 0$$

Summary of Classification

Systems of conservation laws

(Two independent variables) Consider:

$$\underline{u}_t + \underline{F}(\underline{u},x,t)_x = \underline{P}(\underline{u},x,t)$$

Let $A_{ij} = \dfrac{\partial F_i}{\partial u_j}$

$$\underline{u}_t + \underline{\underline{A}}(\underline{u},x,t)\cdot\underline{u}_x = \underbrace{\underline{P}(\underline{u},x,t) - \left.\frac{\partial \underline{F}}{\partial x}\right|_{u,t \text{ fixed}}}_{\underline{Q}(\underline{u},x,t)}$$

Left eigenvectors:

Suppose $\underline{\ell}_i^T \cdot \underline{\underline{A}} = \lambda_i \underline{\ell}_i^T$.

$\lambda_i = \lambda_i(\underline{u},x,t)$ and $\ell_i = \ell_i(\underline{u},x,t)$.

Get: $\underline{\ell}_i^T \cdot (\underline{u}_t + \lambda_i \underline{u}_x) = \underline{\ell}_i^T \cdot \underline{Q}$

Characteristics:

$$\frac{dX_i}{dt} = \lambda_i \qquad \underline{\ell}_i^T \cdot \frac{d\underline{u}}{dt} = \underline{\ell}_i^T \cdot \underline{Q}$$

Hyperbolic if there is a complete set of real eigenvalues and eigenvectors.

Right eigenvectors (Leveque §2.9, §9.8)

Suppose $\underline{\underline{A}}\cdot\underline{\underline{R}} = \underline{\underline{R}}\cdot\underline{\underline{\Lambda}}$

Assume hyperbolic.

So Λ real and R invertible.

Get $\underline{\underline{A}} = \underline{\underline{R}}\cdot\underline{\underline{\Lambda}}\cdot\underline{\underline{R}}^{-1}$

So $\underline{u}_t + \underline{\underline{R}}\cdot\underline{\underline{\Lambda}}\cdot\underline{\underline{R}}^{-1}\cdot\underline{u}_x = \underline{Q}$

So $\underline{\underline{R}}^{-1}\cdot\underline{u}_t + \underline{\underline{\Lambda}}\cdot\underline{\underline{R}}^{-1}\cdot\underline{u}_x = \underline{\underline{R}}^{-1}\cdot\underline{Q}$

Note: $(\underline{\underline{R}}^{-1}\cdot\underline{u})_x = \underline{\underline{R}}^{-1}_x\cdot\underline{u} + \underline{\underline{R}}^{-1}\cdot\underline{u}_x$

So $$(\underline{\underline{R}}^{-1}\cdot\underline{u})_t + \underline{\underline{\Lambda}}\cdot(\underline{\underline{R}}^{-1}\cdot\underline{u})_x = \underbrace{\left(\underline{\underline{R}}^{-1}\cdot\underline{Q} + \underline{\underline{\Lambda}}\cdot\underline{\underline{R}}^{-1}_x\cdot\underline{u} + \underline{\underline{R}}^{-1}_t\cdot\underline{u}\right)}_{\underline{Q}^*(\underline{u},x,t)}$$

Let $\underline{w} = \underline{\underline{R}}^{-1}\cdot\underline{u}$

(so $\underline{\underline{R}}\cdot\underline{w} = \underline{u}$)

Get:
$$\underline{w}_t + \underline{\underline{\Lambda}}\cdot\underline{w}_x = \underbrace{\left(\underline{\underline{R}}^{-1}\cdot\underline{Q} + \underline{\underline{\Lambda}}\cdot\underline{\underline{R}}^{-1}_x\cdot\underline{\underline{R}}\cdot\underline{w} + \underline{\underline{R}}^{-1}_t\cdot\underline{\underline{R}}\cdot\underline{w}\right)}_{\underline{\tilde{Q}}(\underline{w},x,t)}$$

Characteristics

$$\frac{dX_i}{dt} = \lambda_i$$

Concern: $\underline{\underline{\Lambda}}$ depends on $\underline{\underline{A}}$, which depends on $\underline{u}$, which depends on $\underline{w}$ and $\underline{\underline{R}}$. But $\underline{\underline{R}}$ depends on $\underline{\underline{A}}$. Implicit.

$$\frac{d}{dt} w_i(X_i(t),t) = \tilde{Q}_i(\underline{w}(X_i(t),t), X_i(t), t)$$

Classification

$$\underline{u}_t + \underline{\underline{A}}\cdot\underline{u}_x = \underline{Q}$$

- Hyperbolic if λ's real, complete set of ℓ's
- Parabolic if λ's real, not complete set of ℓ's
- Elliptic if λ's complex.

Classification of 2^{nd} order quasilinear PDEs in 2 independent variables

Characteristic via left eigenvectors (again)
(See Morton & Mayers p 86)

(I) $\underline{u}_t + \underline{\underline{A}}(u) \cdot \underline{u}_x = 0$

Suppose $LA = \Lambda L$

$\underline{\underline{L}} \cdot \underline{u}_t + \underline{\underline{\Lambda}} \cdot \underline{\underline{L}} \cdot u_x = 0$

$$\Big(\sum_j L_{ij} \cdot \underbrace{(\partial_t \underline{u} + \lambda_i \partial_x \underline{u})}_{\frac{d\underline{u}}{dt} \text{ if } \lambda_i = \frac{dX}{dt}.}\Big)_j = 0\Big)$$

Characteristic equations:

$$\frac{dX_i}{dt} = \lambda_i$$

$$\sum_j L_{ij} \left(\frac{d^i}{dt}\underline{u}\right)_j = 0 \quad \left(\text{i.e. } \ell_i^T \cdot \frac{d^i}{dt} \underline{u} = 0\right)$$

(II) Suppose Riemann invariants $\underline{r} = \underline{r}(\underline{u})$ exist such that

$$\underline{r}_t = \underline{\underline{L}} \cdot \underline{u}_t \text{ and } \underline{r}_x = \underline{\underline{L}} \cdot \underline{u}_x$$

So $\underline{r}_t + \underline{\underline{\Lambda}} \cdot \underline{r}_x = 0 \quad \left((\partial_t r)_i + \lambda_i (\partial_x r)_i = 0\right)$

"Riemann invariants can always be defined for a system of two equations." (How?)

Characteristic equations: ($L_{ij} = \delta_{ij}$ above)

$$\frac{dX_i}{dt} = \lambda_i$$

$$\frac{d^i}{dt} \underline{r}_i = 0$$

Understanding Characteristics and shocks (SWE)

Riemann invariants:

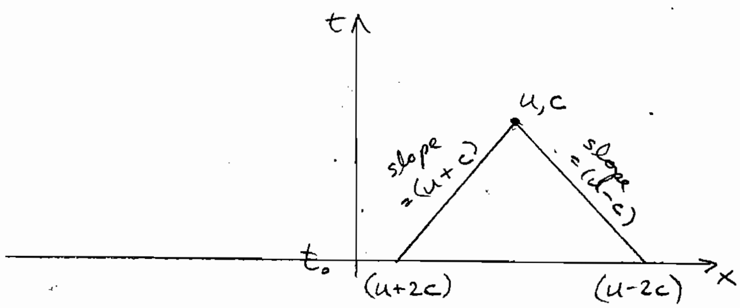
$$\frac{dX^+}{dt} = u + c \qquad \frac{d^+(u+2c)}{dt} = 0$$

$$\frac{dX^-}{dt} = u - c \qquad \frac{d^-(u-2c)}{dt} = 0$$

Shock conditions:

$$[uh] = \dot{S}[h]$$

$$[u^2h + gh^2/2] = \dot{S}[uh]$$

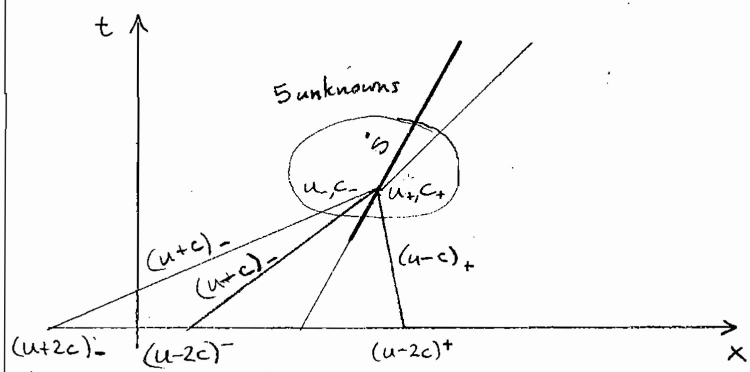


When there is no shock, for an infinitesimal time interval, there are two unknowns (u and c) and two equations, one for each Riemann invariant:

$$[u+2c]_{x,t} \approx [u+2c]_{x-(u+c)(t-t_0),\, t_0}$$

$$[u-2c]_{x,t} \approx [u-2c]_{x+(u+c)(t-t_0),\, t_0}$$

Case of a shock:



Recall: $c = \sqrt{gh}$

So $h = \frac{c^2}{g}$

The shock conditions become:

$$[uc^2] = \dot{S}[c^2]$$

$$[u^2c^2 + c^4] = \dot{S}[uc^2]$$

Assuming that the plus characteristics are passing through the shock and the minus characteristics are impinging on the shock, the Riemann invariants give us 3 more equations.

So we get 5 equations in 5 unknowns.