Wronskian determinant

$\boxed{\text{Let } X'(t) = P(t)\,X(t)}$

where X and P are $n \times n$ matrices.

So $X_i^{k\,\prime} = \sum_j P_i^j X_j^k$

where $X_i^k = i^{th}$ row, k^{th} column of X.

Let $W = \det X$.

So $W = \sum_\sigma (-1)^\sigma \prod_\ell X_\ell^{\sigma(\ell)}$

So
$$W' = \sum_\sigma (-1)^\sigma \sum_r X_r^{\sigma(r)\,\prime} \prod_{\ell \neq r} X_\ell^{\sigma(\ell)}$$
$$= \sum_\sigma (-1)^\sigma \sum_r \sum_j P_r^j X_j^{\sigma(r)} \prod_{\ell \neq r} X_\ell^{\sigma(\ell)}$$
$$= \sum_r \sum_j P_r^j \underbrace{\sum_\sigma (-1)^\sigma X_j^{\sigma(r)} \prod_{\ell \neq r} X_\ell^{\sigma(\ell)}}_{\substack{0 \text{ if } j \neq r, \\ W \text{ if } j = r}}$$
$$= \left(\sum_r P_r^r\right) W$$
$$= (\operatorname{tr} P)\, W.$$

i.e.:

$$\boxed{(\det X)' = (\operatorname{tr} P)(\det X)}$$

Linear Independence & the Wronskian

Thm Let $Y(t)$ be a differentiable matrix, and let $I=(a,b)$ be an interval of t on which $W:=\det Y$ (the Wronskian) is nonzero.
Then $\exists P(t)$ s.t. $Y' = PY$ on I.

Pf Let $P = Y'Y^{-1}$. Done.

Thm Let $y_j(t)$, $j=0..(n-1)$ be a set of n functions each of which is n times differentiable.
Let $Y = \begin{bmatrix} y_0^{(n-1)} & \cdots & y_0' & y_0 \\ \vdots & & & \\ y_{n-1}^{(n-1)} & \cdots & y_{n-1}' & y_{n-1} \end{bmatrix}$,
the $n\times n$ Wronskian matrix.
Let $W = \det Y$, the Wronskian determinant
Let I be an interval of t on which $W(t) \neq 0$.
Then the y_j constitute a fundamental set of solutions on this interval for some ODE.

Pf We seek a linear operator
$L[y] = y^{(n)} - p_0(t) y^{(n-1)} - \cdots - p_{n-1}(t) y$ such that
$L[y_j] = 0$, $j = 0..(n-1)$, i.e.

$$\underbrace{\begin{pmatrix} y_0^{(n)} \\ \vdots \\ y_{n-1}^{(n)} \end{pmatrix}}_{\text{Call } \vec{y}^{(n)}} = \underbrace{\begin{bmatrix} y_0^{(n-1)} & \cdots & y_0 \\ \vdots & & \\ y_{n-1}^{(n-1)} & \cdots & y_{n-1} \end{bmatrix}}_{\text{Call } Y.} \underbrace{\begin{bmatrix} p_0 \\ \vdots \\ p_{n-1} \end{bmatrix}}_{\text{Call } \vec{p}}.$$

Since $\det Y \neq 0$, solve for $\vec{p}$:
$\vec{p} = Y^{-1} \vec{y}^{(n)}$. Done.

In fact, Cramer's rule lets us write down an explicit formula for $\vec{p}$

$$p_j = \frac{\begin{vmatrix} y_0^{(n-1)} \cdots y_0^{(j+1)} & y_0^{(n)} & y_0^{(j-1)} \cdots y_0 \\ & \cdots & \\ y_{n-1}^{(n-1)} \cdots y_{n-1}^{(j+1)} & y_{n-1}^{(n)} & y_{n-1}^{(j-1)} \cdots y_0 \end{vmatrix}}{W}$$

The ODE is: $[y^{(n)} \cdots y^{(0)}] \begin{bmatrix} p_0 \\ \vdots \\ p_n \end{bmatrix} = 0$

We can put these two equations together.
The ODE which our set of y_j all satisfy:

$$\frac{\begin{vmatrix} y^{(n)} & \cdots & y \\ y_0^{(n)} & \cdots & y_0 \\ \vdots & & \vdots \\ y_n^{(n)} & \cdots & y_n \end{vmatrix}}{W} = 0$$

The coefficients match with p_j above.
More simply, one may trivially verify that each $\vec{y}_j(t)$ satisfies this ODE and that the leading coefficient is 1 (due to W in the denominator).

Matrix exponential

Def $\Phi(t) = \exp(A,t)$ if Φ satisfies:
$$\Phi' = A\Phi, \text{ and } \Phi(0) = I.$$

Thm Let $\Phi_A(t) = \exp(A,t)$, and
Let $\Phi_B(t) = \exp(B,t)$.
If $At_1 = Bt_2 = C$ $\rightarrow$
then $\Phi_A(t_1) = \Phi_B(t_2)$

Pf $\Phi_A' = A\Phi_A$ and $\Phi_B' = B\Phi_B$
$\Phi_A(0) = I$ $\qquad \Phi_B(0) = I$

So $t_1\Phi_A' = \underbrace{(t_1 A)}_{C}\Phi_A$ and $t_2\Phi_B' = \underbrace{(t_2 B)}_{C}\Phi_B$

Let $t = t_1\tau_1$

$$\frac{d\Phi_A}{d\tau_1} = \frac{d\Phi_A}{dt}\frac{dt}{d\tau_1} = \Phi_A' t_1$$

analogously for B.

So $\dfrac{d\Phi_A}{d\tau_1} = C\Phi_A$ and $\dfrac{d\Phi_B}{d\tau_2} = C\Phi_B$

$\Phi_A(\tau_1 = 0) = I$ $\qquad \Phi_B(\tau_2 = 0) = I$

Need $\Phi_A(\tau_1 = 1) = \Phi_B(\tau_2 = 1)$.

Obvious. //

So the matrix exponential is well-defined if we say:

Def $\Phi(t) = \exp(At)$ if Φ satisfies
$$\Phi' = A\Phi \text{ and } \Phi(0) = I.$$

Properties

Thm If $\Phi' = A\Phi$, then Φ is infinitely differentiable.

Pf. Case A invertible:
Write $A^{-1}\Phi' = \Phi$.
So $A^{-1}\Phi'' = \Phi'$.
Continue inductively.

General case.
$$(\varphi')' = (A\varphi)' = A\varphi' = A^2\varphi.$$
Continue inductively.

Properties

Thm $\underbrace{\exp(T^{-1}ATt)}_{\text{Call } \Psi} = T^{-1}\underbrace{\exp(At)}_{\text{Call } \Phi}T$

Pf

So $\Psi' = T^{-1}AT\Psi$ $\qquad \Phi' = A\Phi$
$\Psi(0) = I$ $\qquad \Phi(0) = I$

Need $\Psi = T^{-1}\Phi T$

Well $(T\Psi)' = A(T\Psi)$

$(T\Psi)\big|_{t=0} = T$

So $T\Psi = \Phi T$

So $\Psi = T^{-1}\Phi T$. //

Thm If $AB = BA$,
then $\underbrace{\exp((A+B)t)}_{\text{Call } \Phi_{A+B}} = \underbrace{\exp(At)}_{\text{Call } \Phi_A}\underbrace{\exp(Bt)}_{\text{Call } \Phi_B}$

Pf

Need $\Phi_{A+B} = \Phi_A\Phi_B$

Show that both sides satisfy the same differential equation.
Both sides are initially I.

lhs$'$ = $(A+B)$ lhs

rhs$'$ = $\Phi_A'\Phi_B + \Phi_A\Phi_B'$
$= A\Phi_A\Phi_B + \underbrace{\Phi_A B}_{\text{Need} = B\Phi_A}\Phi_B'$.

So we are done if we can prove the following lemma:

Lemma If $AB = BA$,
then $\underbrace{\exp(At)B}_{\text{Call } \Psi} = B\underbrace{\exp(At)}_{\text{Call } \Phi_A}$

COR: $Ae^{At} = e^{At}A$

Pf Show that both sides satisfy the same differential equation.

$\Psi = \Phi_A B$ is solution to:
$$\Psi' = A\Psi, \quad \Psi(0) = B.$$

Likewise,
$$(B\Phi_A)' = B\Phi_A' = BA\Phi_A = A(B\Phi_A),$$
and $(B\Phi_A)\big|_{t=0} = B$ //

(matrix exponential)

Power Series definition

$$\exp(At) \equiv \lim_{m\to\infty} S_m(t)$$

where $S_m(t) = \sum_{k=0}^{m} \frac{A^k t^k}{k!}$

Convergence

Let $\|\cdot\|$ denote a norm on $\mathbb{C}^{n\times n}$ such that $\|AB\| \le \|A\|\cdot\|B\|$

(e.g. $\|A\| = \sup_{\|x\|=1} \|Ax\|$)

Let $\|\cdot\|_{max}$ denote the magnitude of the largest entry.

These norms are equivalent.

So write $\left.\begin{array}{l}\|A\| \le C_2 \|A\|_{max} \\ \|A\|_{max} \le C_1 \|A\|\end{array}\right\} \forall A \in \mathbb{C}^{n\times n}$

Fix $A \in \mathbb{C}^{n\times n}$.

$$\|A^k\|_{max} \le C_1 \|A^k\| \le C_1 \|A\|^k$$

The power series defining $\exp(At)$ is a power series in each matrix component, and therefore it is convergent, continuous, differentiable, and analytic if the power series for each component is.

The power series of a component is analytic if its coefficients are bounded by the coefficients of the power series of an exponential.

Indeed,

$$\exp(At)_{ij} = \sum_{k=0}^{\infty} \frac{(A^k)_{ij} t^k}{k!} \le \sum_{k=0}^{\infty} \frac{\|A^k\|_{max} t^k}{k!} \le \sum_{k=0}^{\infty} \frac{C_1 \|A\|^k t^k}{k!},$$

whose terms are indeed (bounded by) the terms of

$$C_1 \exp(\|A\| t).$$

So $\exp(At)$ is analytic.

Thm $d_t \exp(At) = A \exp(At)$.

Pf The theory of analytic functions defined by power series may be applied componentwise to yield this result. Indeed,

$$d_t \exp(At) = d_t \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} d_t \frac{A^k t^k}{k!} = \sum_{k=1}^{\infty} A^k \frac{t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} A \frac{A^k t^k}{k!}$$

[The i-j^{th} component is:

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{n} A_{i\ell} \frac{(A^k)_{\ell j} t^k}{k!} = \sum_{\ell=1}^{n} A_{i\ell} \sum_{k=0}^{\infty} \frac{(A^k)_{\ell j} t^k}{k!}$$]

$$= A \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = A \exp(At).$$

Since $\exp(A\cdot 0) = I$, $\exp(At)$ satisfies the differential equation on the obverse, so the definitions are equivalent.

Note that the theorems on the right hand column of the obverse can be proved easily using the power series definition.

Forced 1st-order linear systems

Problem: $\underline{x}' = \underline{\underline{P}}(t)\cdot\underline{x} + \underline{g}(t)$

Method of variation of parameters

Suppose $\boxed{\underline{\underline{\Psi}}' = \underline{\underline{P}}(t)\cdot\underline{\underline{\Psi}},\ \det \Psi_0 \neq 0.}$

(i.e., $\underline{\underline{\Psi}}$ is a fundamental matrix (whose columns are solutions) of the unforced system.)

Guess $\underline{x}(t) = \underline{\underline{\Psi}}\cdot\underline{u}(t)$, where $\underline{u}(t)$ is an unknown function.

Then $x' = \Psi' u + \Psi u'$

So $\cancel{\Psi' u} + \Psi u' = \cancel{P\Psi u} + g$

So $u' = \Psi^{-1} g$, since $\det \Psi \neq 0\ \forall t$, since $(\det\Psi)' = (\operatorname{tr} P)(\det\Psi)$ and $(\det \Psi_0) \neq 0$.

So $u = \underline{c} + \int \Psi^{-1} g$

So $\boxed{\underline{x} = \underline{\underline{\Psi}}\cdot(\underline{c} + \int \underline{\underline{\Psi}}^{-1}\cdot \underline{g})}$

$\underline{x} = \Psi\,(c + \int_0^t \Psi^{-1} g)$

$\underline{x}_0 = \Psi_0 c$

$c = \Psi_0^{-1} x_0$

$\boxed{\underline{x} = \underline{\underline{\Psi}}\cdot(\underline{\underline{\Psi}}_0^{-1}\underline{x}_0 + \int_0^t \underline{\underline{\Psi}}\cdot \underline{g})}$

Method of integrating factor

Let $\boxed{\tilde{\underline{\underline{P}}} = -\underline{\underline{P}}.}$

$\underline{x}' + \tilde{\underline{\underline{P}}}(t)\cdot\underline{x} = \underline{g}$

Want to multiply both sides on the left by an integrating factor $\underline{\underline{M}}(t)$ so that on the left side we get:

$\underline{\underline{M}}\cdot[\underline{x}' + \tilde{\underline{\underline{P}}}\cdot\underline{x}] = [\underline{\underline{M}}\cdot\underline{x}]'$

i.e. $Mx' + M\tilde{P}x = Mx' + M'x$

Need $\boxed{M' = M\tilde{P}}$ Any such M s.t. $\det M_0 \neq 0$ will suffice.

i.e. $(M^T)' = \tilde{P}^T M^T$

So $[Mx]' = Mg$

So $\boxed{x = \underline{\underline{M}}^{-1}\cdot(\underline{c} + \int \underline{\underline{M}}\cdot \underline{g})}$

$\underline{x} = M^{-1}(c + \int_0^t Mg)$

$x_0 = M^{-1} c$

$c = M x_0$

$\boxed{\underline{x} = \underline{\underline{M}}^{-1}\cdot(\underline{\underline{M}}\cdot\underline{x}_0 + \int_0^t \underline{\underline{M}}\cdot\underline{g})}$

How do Ψ and M relate?

The case $g = 0$ makes clear that M^{-1} is a fundamental matrix.

Claim: $(\Psi^{-1})' = (\Psi^{-1})(-P)$

Pf: $(\Psi\Psi^{-1})' = \underline{0}$.

So $\Psi'\Psi^{-1} + \Psi(\Psi^{-1})' = \underline{0}$.

But $\Psi' = P\Psi$.

So $(\Psi^{-1})' = (\Psi^{-1})(-P)$

So $\boxed{M = \Psi^{-1}}$

(Interesting result:

Let $\Psi' = P\Psi$

then $(\Psi^{-T})' = -P^T(\Psi^{-T})$

Note that if P is constant and if Ψ_0 commutes with P, then $P\Psi(t) = \Psi(t)P\ \forall t$.)

Constant-coefficient forced ODE

Problem: $\boxed{\underline{x}' = \underline{\underline{A}} \cdot \underline{x} + \underline{g}(t)}$ (*)

($\underline{\underline{A}}$ is constant). $\underline{x}(0) = \underline{x}_0$

Method of diagonalization

Let V be a matrix whose columns constitute a full set of generalized eigenvectors, and let Λ be the corresponding matrix of elementary diagonal eigenvalue blocks:

$$AV = V\Lambda.$$

Let $\underline{x} = \underline{\underline{V}} \cdot \underline{y}$

We can diagonalize the ODE by:

① multiplying both sides of (*) on the left by V^{-1}, and

② replacing:

ⓐ A with $V\Lambda V^{-1}$, or

ⓑ $V^{-1}A$ with ΛV^{-1}, or

ⓒ AV with $V\Lambda$ after doing the subsequent step:

③ replacing

ⓐ x with Vy, or

ⓑ $V^{-1}x$ with y.

Get:

$$(V^{-1}x)' = \Lambda (V^{-1}x) + V^{-1}g$$

So $\boxed{y' = \Lambda y + V^{-1} g.}$

This is sufficiently decoupled to solve.

Then transform back using

$\boxed{x = \underline{\underline{V}} \cdot y}$

Method of Laplace Transform

Let $\mathcal{L}$ = Laplace transform.

Apply $\mathcal{L}$ to both sides.

Write $\underline{X} = \mathcal{L}\{\underline{x}\}$,

$\underline{G} = \mathcal{L}\{\underline{g}\}$

Recall $\mathcal{L}\{x'\} = s\underline{X} - \underline{x}_0$

So $s\underline{X} - \underline{x}_0 = \underline{\underline{A}} \cdot \underline{X} + \underline{G}$

$$(Is - A)X = G + x_0$$

$$X = (Is - A)^{-1}(G + x_0)$$

$$x = \mathcal{L}^{-1}\{(Is - A)^{-1}(G + x_0)\}$$

Method of Undetermined Coefficients

Suppose $\underline{g}(t)$ is of the form

$$\underline{g}(t) = e^{\alpha t}(P_n(t)\cos\beta t + Q_n(t)\sin\beta t)$$

where P_n and Q_n are n^{th}-order polynomials in t with constant vector coefficients.

Then guess a solution of the form

$$\underline{x}(t) = t^s e^{\alpha t}(\tilde{\underline{A}}(t)\cos\beta t + \tilde{\underline{B}}(t)\sin\beta t)$$

where $\tilde{A}(t)$ and $\tilde{B}(t)$ are n^{th}-order polynomials in t with undetermined constant vector coefficients, and s is the number of times $\alpha + i\beta$ is a root of the characteristic polynomial of the matrix $\underline{\underline{A}}$.

(Need to prove that there is a solution of this form.)