Solving Maxwell in a Vacuum

	Maxwell homogeneous	forced
constraint	$\nabla\cdot B = 0$	$\nabla\cdot E = \rho$
evolution	$\nabla\times E = -B_t$	$\nabla\times B = E_t + J$

Potentials

Write $\boxed{B = \nabla\times A}$

So $\nabla\times E = -\nabla\times A_t$

i.e. $\nabla\times(E + A_t) = 0$

Write $E + A_t = -\nabla\varphi$

i.e. $\boxed{E = -\nabla\varphi - A_t}$

Gauge freedom

Let $B = \nabla\times A'$

$E = -\nabla\varphi' - A'_t$

$\nabla\times(A' - A) = 0$

$A' - A = \nabla\lambda \Rightarrow \boxed{A' = A + \nabla\lambda}$

$\nabla(\varphi' - \varphi) + (A' - A)_t = 0$

$\nabla(\varphi' - \varphi) + \nabla\lambda_t = 0$

So $\nabla(\varphi' - \varphi + \lambda_t) = 0$

So $\varphi' - \varphi + \lambda_t = k(t)$.

Absorb k into λ_t.

So $\boxed{\varphi' = \varphi - \lambda_t}$

Solving via potentials

$$\boxed{-\rho = \nabla^2\varphi + (\nabla\cdot A)_t}$$

$$\nabla\times\nabla\times A = -\nabla\varphi_t - A_{tt} + J$$

$$\boxed{A_{tt} = \nabla^2 A - \nabla(\nabla\cdot A + \varphi_t) + J}$$

Check that any such A, φ satisfy Maxwell:

$\nabla\cdot B = \nabla\cdot\nabla\times A = 0$ ✓

$\nabla\cdot E = -\nabla\cdot(\nabla\varphi + A_t) = \rho$ ✓

$\nabla\times E = -(\nabla\times A)_t = -B_t$ ✓

$$\begin{aligned}\nabla\times B &= \nabla\times\nabla\times A \\ &= \nabla\nabla\cdot A - \nabla^2 A \\ &= J - A_{tt} - \nabla V_t \\ &= J + E_t \quad ✓\end{aligned}$$

Ability to prescribe $\nabla\cdot A$

Claim that a vector field A that decays at ∞ is uniquely determined by its curl and divergence.

Given: B, D,

$$(\dagger)\quad \begin{cases}\nabla\times A = B \\ \nabla\cdot A = D\end{cases}$$

Find A.

Seek $A = \nabla\times\alpha + \nabla\beta$

$\boxed{\nabla^2\beta = D}$

$\nabla\times\nabla\times\alpha = B$

$\underbrace{\nabla(\nabla\cdot\alpha)}_{\text{Set } 0} - \nabla^2\alpha = B$

$\boxed{\nabla^2\alpha = -B}$

Such α and β are unique, and for such α and β, $\nabla\times A = B$ and $\nabla\cdot A = D$.

uniqueness: A, A' two solutions to $(\dagger)$

$\nabla\times(A' - A) = 0 \Rightarrow A' - A = \nabla\lambda \quad \exists\lambda\to 0$ at ∞

$\nabla\cdot(A' - A) = 0 \Rightarrow \nabla^2\lambda = 0 \Rightarrow \lambda = 0$.

Generic Gauge

Require $\boxed{\nabla\cdot A = D}$. So: (assume D is a divergence)

$$(\dagger)\quad \begin{cases}\nabla^2\varphi = -\rho - D_t \\ A_{tt} = \nabla^2 A - \nabla(D + \varphi_t) + J.\end{cases}$$

Information needed: $B_0, E_0, \rho_0, J(t)$

Computed information (ICs)

$\rho_t + \nabla\cdot J = 0 \to \rho(t)$

$\left.\begin{aligned}\nabla\times A &= B \\ \nabla\cdot A &= D\end{aligned}\right\} \to A_0$, since B_0, D_0 are known.

$\left.\begin{aligned}\nabla\times A_t &= -\nabla\times E \\ \nabla\cdot A_t &= D_t\end{aligned}\right\} \to (A_t)_0$ since $E_0, (D_t)_0$ are known.

Drift from gauge condition (due to accumulated numerical error)

Assume $(\dagger)$ holds. (but ICs have diverged) Then

$$(\nabla\cdot A)_{tt} = \nabla^2(\nabla\cdot A) - \nabla^2 D + \underbrace{-\nabla^2\varphi_t + \nabla\cdot J}_{D_{tt} + \underbrace{\rho_t + \nabla\cdot J}_{0}}$$

$$\boxed{(\nabla\cdot A - D)_{tt} = \nabla^2(\nabla\cdot A - D)}$$

error obeys wave equation.

Coulomb gauge: $D = 0$.

Lorentz gauge: $D = -\varphi_t$. So $D_t = -\varphi_{tt}$.

Freedom to choose $\varphi_0, (\varphi_t)_0$.

Wave equation $\varphi_{tt} = \nabla^2\varphi + \rho$ determines $\varphi(t)$

Solution remains physical even if it diverges from true?

No.

(each gauge in detail)

Coulomb gauge

Require $\nabla\cdot A = 0$

Potential evolution equations

$$(\dagger)\quad \begin{cases} \nabla^2\varphi = -\rho \\ A_{tt} = \nabla^2 A - \nabla\varphi_t + J \end{cases}$$

Information needed to solve:

$J(t)$
$\rho_0 := \rho(t=0)$. $\Big\}$ Gives $\rho(t)$ via $\rho_t + \nabla\cdot J = 0$.
$E_0 := E(t=0)$
$B_0 := B(t=0)$

Computed information

$\varphi(t)$ comes from $\nabla^2\varphi = -\rho$.

$\left.\begin{matrix}\nabla\times A = B \\ \nabla\cdot A = 0\end{matrix}\right\} \Rightarrow A_0$, since B_0 is known.

$\left.\begin{matrix}\nabla\times A_t = -\nabla\times E \\ \nabla\cdot A_t = 0\end{matrix}\right\} \Rightarrow (A_t)|_{t=0}$ since E_0 is known

Drift from gauge condition
(due to accumulated numerical error)

Assume $(\dagger)$. Then

$$\boxed{(\nabla\cdot A)_{tt} = \nabla^2(\nabla\cdot A)} + \underbrace{(-\nabla^2\varphi_t + \nabla\cdot J)}_{\rho_t + \nabla\cdot J = 0}$$

So the error $\nabla\cdot A$ obeys the wave equation.

Is this a problem since electromagnetic disturbances also propagate at this speed?

Lorentz gauge

Require $\nabla\cdot A + \varphi_t = 0$

Potential evolution equations

$$(\dagger)\quad \begin{cases} \varphi_{tt} = \nabla^2\varphi + \rho \\ A_{tt} = \nabla^2 A + J \end{cases}$$

$\nabla\cdot E = \rho$
$\nabla\times B = E_t + J$

Information needed
(same)
$J(t), \rho_0, E_0, B_0$

Computed information

$\rho(t)$ from $\rho_t + \nabla\cdot J = 0$

$\left.\begin{matrix}\nabla\times A = B \\ \nabla\cdot A = -\varphi_t\end{matrix}\right\} \Rightarrow A_0$ per choice of $(\varphi_t)_0$, since B_0 is known

$\left.\begin{matrix}(\nabla\times A)_t = -\nabla\times E \\ \nabla\cdot A_t = -\nabla^2\varphi - \rho\end{matrix}\right\} \Rightarrow (A_t)_0$ per choice of φ_0, since E_0, ρ_0 are known

Verification of freedom of initial choice of φ_0 (equivalently $(\nabla\cdot A_t)_0$) and of $(\varphi_t)_0$ (equivalently $(\nabla\cdot A_0)$)

Recall freedom of gauge transformation:
$A' = A + \nabla\lambda$
$\varphi' = \varphi - \lambda_t$

Assume A, φ are Maxwell potentials
(not necessarily satisfying Lorentz gauge condition).

Want A' and φ' to satisfy Lorentz

Want $\nabla\cdot A' + \varphi'_t = 0$

$$\nabla\cdot A + \nabla^2\lambda + \varphi_t - \lambda_{tt} = 0$$

$$\boxed{\lambda_{tt} = \nabla^2\lambda + (\nabla\cdot A + \varphi_t)}$$

Freedom to choose $\lambda_0, (\lambda_t)_0$
$\Rightarrow$ freedom to specify $(\lambda_t)_0, (\lambda_{tt})_0$
$\Rightarrow$ freedom to specify $\varphi_0, (\varphi_t)_0$

Drift from gauge condition

Assume $(\dagger)$. Then

$$(\nabla\cdot A)_{tt} = \nabla^2(\nabla\cdot A) + \underbrace{\nabla\cdot J}_{= -\rho_t = \nabla^2\varphi_t - \varphi_{ttt}}$$

i.e.

$$\boxed{(\nabla\cdot A + \varphi_t)_{tt} = \nabla^2(\nabla\cdot A + \varphi_t)}$$

same issues