Maxwell's Equations inside Matter

(Griffiths p309 — better argument?)

$J_p :=$ current due to motion of bound charges ($=$: polarization current).

$J_p \cdot \hat{n}$ = rate of change of bound surface charge density on small slab with normal $\hat{n}$.

$$= (P \cdot \hat{n})_t$$

$$= P_t \cdot \hat{n}$$

Can also get this by time-stepping, alternately letting each molecules move and deform.

So $\boxed{J_p = \partial_t P}$

Recall vacuum equations:

$$\nabla \times E = -B_t \qquad \nabla \cdot B = 0$$

$$\nabla \times \left(\frac{B}{\mu_0}\right) = J + (\varepsilon_0 E)_t \qquad \nabla \cdot (\varepsilon_0 E) = \rho$$

$$c^2 \mu_0 \varepsilon_0 = 1$$

Recall:

$$\rho = \rho_f + \rho_b$$

$$= \rho_f - \nabla \cdot P$$

So $\nabla \cdot (\underbrace{\varepsilon_0 E + P}_{\text{Call } D}) = \rho_f$

Recall:

$$J = J_f + J_b + J_p$$

$$= J_f + \nabla \times M + \partial_t P$$

So $\nabla \times \big(\underbrace{\frac{B}{\mu_0} - M}_{\text{Call } H}\big) = J_f + (\underbrace{\varepsilon_0 E + P}_{D})_t$

So:

$$\boxed{\begin{array}{l|l} \nabla \times E = -B_t & \nabla \cdot B = 0 \\ \nabla \times H = J_f + D_t & \nabla \cdot D = \rho_f \\ H := \dfrac{B}{\mu_0} - M & \\ D := \varepsilon_0 E + P & \end{array}}$$

(Linear medium) $D = \varepsilon E$ | $P = \chi_e (\varepsilon_0 E)$ $\quad \varepsilon = \varepsilon_0(1+\chi_e)$

$H = \mu^{-1} B$ | $M = \chi_m H$ $\quad (\mu = \mu_0(1+\chi_m))$

Putting everything in terms of primary rather than auxiliary fields:

$$\nabla \times \left(\frac{B}{\mu_0} - \left(\frac{\mu}{\mu_0} - 1\right)\mu^{-1} B\right) = J_f + (\varepsilon E)_t$$

$$\boxed{\nabla \times \left[\left(\frac{1}{\mu_0} - \left(\frac{\mu}{\mu_0} - 1\right)\mu^{-1}\right) B\right] = J_f + (\varepsilon E)_t}$$

$$\boxed{\nabla \cdot (\varepsilon E) = \rho_f}$$

nondimensional units

$$\nabla \times E = -(cB)_{(ct)} \qquad \nabla \cdot (cB) = 0$$

$$\nabla \times \left(\frac{H}{c\varepsilon_0}\right) = \left(\frac{J_f}{c\varepsilon_0}\right) + \left(\frac{D}{\varepsilon_0}\right)_{(ct)} \qquad \nabla \cdot \left(\frac{D}{\varepsilon_0}\right) = \left(\frac{\rho_f}{\varepsilon_0}\right)$$

$$\left(\frac{H}{c\varepsilon_0}\right) := (cB) - \left(\frac{M}{c\varepsilon_0}\right)$$

$$\left(\frac{D}{\varepsilon_0}\right) := E + \left(\frac{P}{\varepsilon_0}\right)$$

Linear medium

$$\left(\frac{D}{\varepsilon_0}\right) = \left(\frac{\varepsilon}{\varepsilon_0}\right) E \qquad \left(\frac{H}{c\varepsilon_0}\right) = \left(\frac{\mu}{\mu_0}\right)^{-1} (cB)$$

$$\left(\frac{P}{\varepsilon_0}\right) = \chi_e E \qquad \left(\frac{M}{c\varepsilon_0}\right) = \chi_m \left(\frac{H}{c\varepsilon_0}\right)$$

$$\left(\frac{\varepsilon}{\varepsilon_0}\right) = 1 + \chi_e \qquad \left(\frac{\mu}{\mu_0}\right) = 1 + \chi_m$$

Heaviside-Lorentz

$$c \nabla \times E = -(cB)_t \qquad \nabla \cdot (cB) = 0$$

$$c \nabla \times \left(\frac{H}{c\varepsilon_0}\right) = \left(\frac{J_f}{\varepsilon_0}\right) + \left(\frac{D}{\varepsilon_0}\right)_t \qquad \nabla \cdot \left(\frac{D}{\varepsilon_0}\right) = \left(\frac{\rho_f}{\varepsilon_0}\right)$$

$$\left(\frac{H}{c\varepsilon_0}\right) := (cB) + \frac{1}{c}\left(\frac{M}{\varepsilon_0}\right)$$

$$\left(\frac{D}{\varepsilon_0}\right) := E + \left(\frac{P}{\varepsilon_0}\right)$$

Linear Medium (same except:)

$$\left(\frac{1}{c}\right)\left(\frac{M}{\varepsilon_0}\right) = \chi_m \left(\frac{H}{c\varepsilon_0}\right)$$

Electric dipole moment

Assume electrostatics:

$$\nabla\cdot E = \rho$$
$$\nabla\wedge E = 0$$

Invoke a scalar potential:

$$E = -\nabla\varphi$$
$$\nabla^2\varphi = -\rho$$

Solve for φ:

$\varphi = \mathcal{G}(\rho)$, the Newtonian potential of ρ, i.e.

$$\varphi(x) = \frac{1}{4\pi}\int_{x'}\frac{\rho(x')}{|x-x'|} = \frac{1}{4\pi}\int\frac{\rho}{\mathfrak{r}}$$

Taylor expand the Green's function kernel:

$$\frac{1}{|x-x'|} = \frac{1}{|x|} + \frac{x\cdot x'}{|x|^3} + \dots$$

$$\varphi(x) = \frac{1}{4\pi}\left[\frac{1}{|x|}\int\rho + \frac{1}{|x|^3}x\cdot\int x'\rho + \dots\right]$$

× [Assume overall charge neutrality: $\int\rho = 0$]

Define the electric dipole moment by

$$\boxed{p := \int x'\rho(x')\,dx'}$$

p produced by stuff near r $= \int_{r'}(r'-r)\rho(r') = -\int\mathfrak{r}\rho$

Then the dipole potential is:

$$\boxed{\varphi_{dip}(x) = \frac{1}{4\pi}\frac{\hat{x}\cdot p}{|x|^2}}$$

Potential for a dipole located at r' is: $\varphi_{dip}(r) = \frac{1}{4\pi}\frac{\hat{\mathfrak{r}}\cdot p}{\mathfrak{r}^2}$

The dipole field is then

$$E_{dip} = -\nabla\varphi_{dip}.$$

$$\nabla\left(\frac{x\cdot p}{|x|^3}\right) = p\wedge\underbrace{\nabla\wedge\left(\frac{x}{|x|^3}\right)}_{\text{Claim } 0} + p\cdot\nabla\left(\frac{x}{|x|^3}\right)$$

indeed, for any scalar function $f(|x|)$,

$$\nabla\wedge(f\,x) = (\nabla f)\wedge x + f\,\nabla\wedge x$$
$$= f'(|x|)\underbrace{\hat{x}\wedge x}_{0} + f\,\underbrace{\varepsilon_{ijk}\underbrace{\partial_j x_k}_{\delta_{jk}}}_{0}$$
$$= 0$$

$$\nabla\left(\frac{x}{|x|^3}\right) = \frac{\underline{\underline{I}} - 3\hat{x}\hat{x}}{|x|^3}$$

as computed on the magnetostatic page.

another way:

$$\nabla[(x\cdot p)|x|^{-3}]$$
$$= |x|^{-3}\nabla(x\cdot p) + (x\cdot p)\nabla(|x|^{-3})$$
$$= |x|^{-3}p + p\cdot x(-3|x|^{-4}\hat{x})$$
$$= \frac{p - 3p\cdot\hat{r}\hat{r}}{|x|^3}$$

So

$$\boxed{E_{dip}(x) = \frac{1}{4\pi}\frac{3p\cdot\hat{x}\hat{x} - p}{|x|^3}}$$

Field of a polarized object

Let P = dipole moment per unit volume.

$$\varphi_{dip}(r) = \frac{1}{4\pi}\int\frac{(r-r')}{|r-r'|^3}\cdot\underbrace{P(r')\,d^3r'}_{dp \text{ located at } r'}$$
$$= \frac{1}{4\pi}\int\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\cdot P(x'), \text{ where}$$

$\mathfrak{r} := x - x'$ points from the integration (source) variable to the free variable

Let $r := x$ | dP

Let $r' := x'$

Let $\nabla' := \frac{\partial}{\partial r'}$

$$\nabla'\left(\frac{1}{\mathfrak{r}}\right) = \frac{\hat{\mathfrak{r}}}{|\mathfrak{r}|^2}.$$

$$\varphi_{dip} = \frac{1}{4\pi}\int P\cdot\nabla'\left(\frac{1}{\mathfrak{r}}\right)$$
$$= \frac{1}{4\pi}\int\left(\nabla'\cdot\left(\frac{P}{\mathfrak{r}}\right) - \frac{\nabla'\cdot P}{\mathfrak{r}}\right)$$
$$= \frac{1}{4\pi}\left[\oint\hat{n}\cdot\frac{P}{\mathfrak{r}} - \int\frac{\nabla'\cdot P}{\mathfrak{r}}\right]$$

surface charge — charge density

Claim: $\sigma_b = \hat{n}\cdot P$, $\rho_b = -\nabla\cdot P$

better arguments in following work

For ρ_b, break the polarized material into small pieces and use an integration region that fully contains all charges.

The contribution of this piece to φ_{dip} is

$$d\varphi_{dip} = \frac{1}{4\pi}\int\frac{d\rho_b}{\mathfrak{r}} = \frac{1}{4\pi}\int\frac{-\nabla'\cdot P}{\mathfrak{r}}$$

So $\langle d\rho_b\rangle = \langle-\nabla'\cdot P\rangle$, where $\langle\cdot\rangle$ denotes average over this small region,

so $\boxed{\rho_b = -\nabla'\cdot P}$

For σ_b, choose a vanishingly small integration region so that the volume integral will vanish since it is of a higher order of smallness. [Not needed].

Partion the bound charge into a portion that resides in an interior where there is cancelation and a surface with uncancelled charge.

$$\varphi_{dip} = \frac{1}{4\pi}\left[\oint\frac{\sigma_b}{\mathfrak{r}} + \int\frac{\rho_b}{\mathfrak{r}}\right]$$

So $\oint\frac{\sigma_b}{\mathfrak{r}} = \oint\frac{\hat{n}\cdot P}{\mathfrak{r}}$

So $\boxed{\sigma_b = \hat{n}\cdot P}$

Boundary charges of a polarized medium

Imagine a collection of molecules $(M_i)_{i=1}^n$ each located at a point r_i and having a charge distribution which we write $\rho_i(r)$, which is zero except near $r \simeq r_i$.

We determine the long-distance potential φ_i produced by ρ_i.

$$\varphi(r) = \sum_i \frac{1}{4\pi}\int_{\tilde r} \frac{\rho_i(\tilde r)}{|r-\tilde r|}$$

We can assume that $\tilde{\Omega}_i := \tilde r - r_i$ is small. Let $\Omega_i := r - r_i$

$$\frac{1}{|r-\tilde r|} = \frac{1}{|\Omega_i - \tilde\Omega_i|} = \frac{1}{|\Omega_i|} + \frac{\hat\Omega_i\cdot\tilde\Omega_i}{|\Omega_i|^2} + \cdots$$

So

$$\varphi(r) = \frac{1}{4\pi}\sum_i\left(\frac{1}{|\Omega_i|}\int\rho_i + \frac{\hat\Omega_i\cdot}{|\Omega_i|^2}\int\tilde\Omega_i\rho_i + \cdots\right)$$

where $\int\rho_i = i^{th}$ monopole moment, and

$$p_i := \int\tilde\Omega\rho_i := \int_{\tilde r}(\tilde r - r_i)\,\rho_i(\tilde r)$$

is the i^{th} dipole moment.

Let $$\varphi_{dip}(r) = \frac{1}{4\pi}\sum_i \frac{\hat\Omega_i}{|\Omega_i|^2}p_i = \frac{1}{4\pi}\sum_i\int_{r'}\frac{\Omega}{|\Omega|^2}p_i\,\delta^3(r'-r_i)$$

[where $\Omega := r - r'$]

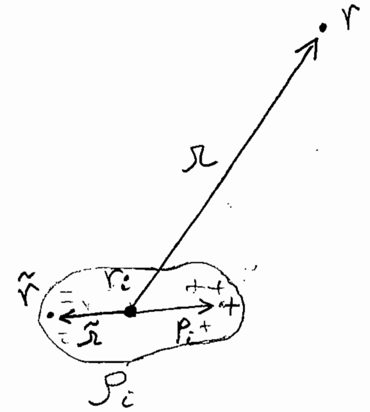
So for an approximation to the identity $f(r) \simeq \delta(r)$,

$$(\varphi_{dip} * f)(r) = \frac{1}{4\pi}\sum_i\int_{r'}\frac{\hat\Omega}{|\Omega|^2}p_i\,f(r'-r_i)$$

Let $P(r) = \sum_i p_i f(r-r_i)$

So $$\int_{r\in dV} P(r)\,d^3r = \sum_i\int_{r\in dV} p_i f(r-r_i) \simeq \sum_{r_i\in dV} p_i$$

$P(r)$ is the <u>dipole moment per unit volume</u>, also called the <u>polarization</u>



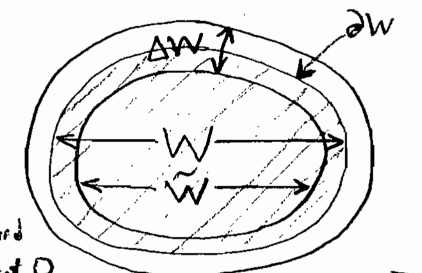
$$\boxed{\varphi_{dip}(r) \simeq (\varphi_{dip} * f)(r) = \frac{1}{4\pi}\int\frac{\hat\Omega}{|\Omega|^2}P(r')\,d^3r' \quad \text{where } \Omega := r - r'}$$

Just as we used an approximation to the identity to smear out the discrete dipoles into a smooth dipole moment per unit volume, so we can use an approximation to the identity to smear out a possibly discontinuous dipole moment per unit volume.

This will allow us to give a clear definition of the surface charge of a polarized continuum with discontinuity at its boundary.

Imagine a polarized medium with polarization P that has support properly within a region W with smooth boundary.

Let $\tilde P = P * f_\varepsilon$, where f_ε is an approximation to the identity whose support is contained in an ε-ball about 0.



[Let $\Delta W =$ "supp $\chi_{\partial W} * B_\varepsilon$" be the region of width ε around ∂W.]

The dipole potential due to P is:

$$\varphi_{dip}(r) = \frac{1}{4\pi}\int_W \underbrace{\frac{\hat\Omega}{|\Omega|^2}P(r')}_{P\cdot\nabla'\left(\frac{1}{\Omega}\right)}\,d^3r' \qquad \text{where } \nabla' := \frac{\partial}{\partial r'}$$

$$\simeq \frac{1}{4\pi}\int_{\tilde W + \Delta W}\tilde P\cdot\nabla'\left(\frac{1}{\Omega}\right)$$

$$= \frac{1}{4\pi}\int_{\tilde W + \Delta W}\left[\nabla'\cdot\left(\frac{P}{\Omega}\right) - \frac{\nabla'\cdot P}{\Omega}\right]d^3r'$$

$$= \frac{1}{4\pi}\left[\oint_{r'\in\partial(\tilde W + \Delta W)}\hat n\cdot\left(\frac{P}{\Omega}\right) - \int_{r'\in\tilde W + \Delta W}\frac{\nabla'\cdot P}{\Omega}\right]$$

(Surface charge of polarized medium)

Again,

$$\varphi_{dip}(r) = \frac{1}{4\pi}\left[\oint_{r' \in \partial(\tilde{W}+\Delta W)} \hat{n}\cdot\left(\frac{P}{\mathcal{r}}\right) d^2r' - \int_{\tilde{W}+\Delta W} \frac{\nabla' \cdot P}{\mathcal{r}} d^3r'\right]$$

We should get approximately the same result for φ_{dip} whether we take

(1) ΔW = the region of width 2ε around ∂W as in the picture, or

(2) $\Delta W = \emptyset$.

Case (2)

Says $\varphi_{dip}(r) = \frac{1}{4\pi}\left[\oint_{\partial\tilde{W}} \hat{n}\cdot\frac{P}{\mathcal{r}} d^2r' - \int_{\tilde{W}} \frac{\nabla' \cdot P}{\mathcal{r}} d^3r'\right]$,

[where $\partial\tilde{W}$ is just inside ∂W].

Case (1)

Since $P = 0$ on $\partial(\tilde{W}+\Delta W)$, says

$$\varphi_{dip}(r) = \frac{1}{4\pi}\left[-\int_{\Delta W} \frac{\nabla' \cdot P}{\mathcal{r}} d^3r' - \int_{\tilde{W}} \frac{\nabla' \cdot P}{\mathcal{r}} d^3r'\right]$$

So we may infer that

$$(*) \quad \boxed{\oint_{r' \in \partial\tilde{W}} \hat{n}\cdot\frac{P}{\mathcal{r}} \simeq \int_{r' \in \Delta W} \frac{-\nabla' \cdot P}{|r-r'|}}$$

See argument on next page. [$-\nabla \cdot P$ is the bound charge density (in ΔW) due to polarization.

I claim that $\hat{n}\cdot\frac{P}{\mathcal{r}}$ is the surface charge density, which is the density of bound charge near ∂W.

To see this, use a partition of unity to write P as a sum of polarizations P_i with small supports.

In this case $\frac{1}{\mathcal{r}} = \frac{1}{|r-r'|}$ in the integrand is roughly constant for a given P_i, so get

$$\sum_i \left(\hat{n}\cdot P_i \cdot \underbrace{(\text{area})}_{\substack{\text{area of surface} \\ \text{where } P_i \\ \text{is nonzero}}} = \underbrace{-\nabla\cdot P_i \cdot (\text{volume})}_{\text{charge in belt } \Delta W \text{ of } P_i} \right)$$

So $\boxed{\hat{n}\cdot P = \text{charge per surface area}}$

Electric Displacement

Assume that the net charge ρ is composed of some free charge ρ_f plus some overall-neutral bound charge ρ_b:

$$\rho = \rho_b + \rho_f$$

The potential caused by the net and free charges is:

$$\varphi = \frac{1}{4\pi}\int \frac{\rho}{\mathscr{r}} \,,\quad \varphi_f = \frac{1}{4\pi}\int \frac{\rho_f}{\mathscr{r}}$$

The potential caused by the bound charge is approximately the potential caused by its dipole moment per unit volume, since we assume that the bound charges come in localized units with net neutral charge:

$$\varphi_b = \frac{1}{4\pi}\int \frac{\rho_b}{\mathscr{r}}$$

$$\simeq \frac{1}{4\pi}\int P\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$$

$$= \frac{-1}{4\pi}\int \frac{\nabla'\cdot P}{\mathscr{r}}$$

So the net potential is:

$$\varphi = \frac{1}{4\pi}\int \left(\rho_f - \nabla'\cdot P\right)\frac{1}{\mathscr{r}}$$

But $\varphi = \frac{1}{4\pi}\int \frac{\rho}{\mathscr{r}}$.

Since φ is uniquely determined from ρ, we have that

$$\rho = \rho_f - \nabla\cdot P$$

i.e. $\boxed{\rho_b = -\nabla\cdot P}$

So: $\nabla\cdot E = \rho$,

$\nabla\cdot E_f = \rho_f$

$\nabla\cdot E_b = \rho_b = -\nabla\cdot P$

Define $\boxed{D := E + P}$ electric displacement

Then $\nabla\cdot D = \nabla\cdot E + \nabla\cdot P$

$= \rho - \rho_b$

i.e. $\boxed{\nabla\cdot D = \rho_f}$

Can we say that $D = E_f$?

No. See Griffiths page 175.

$\nabla\wedge D = \nabla\wedge P \neq 0$ in general.

So $D \neq \frac{1}{4\pi}\int \frac{\rho_f \hat{\mathscr{r}}}{\mathscr{r}^2}$ in general.

Linear medium

Assume $\boxed{P = \chi_e E.}$

Then $D = (1+\chi_e)E$

Define $\boxed{\varepsilon := 1 + \chi_e}$ permittivity or dielectric constant

Then $\boxed{D = \varepsilon E}$

Note $\rho_b = -\nabla\cdot P = -\nabla\cdot(\chi_e E)$

$$= -\nabla\cdot\left(\frac{\chi_e}{1+\chi_e}D\right)$$

$$= -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f + \underbrace{D\,\nabla\cdot\left(\frac{\chi_e}{1+\chi_e}\right)}_{0 \text{ if homogeneous}}$$

0 except at boundary for a conductor.

So it is important to understand boundary charges.

Magnetic dipole moment

Assume magnetostatics:

$$\nabla\cdot B = 0$$
$$\nabla\wedge B = J$$

Invoke the Coulomb potential:

$$\nabla\wedge A = B$$
$$\nabla\cdot A = 0$$

Solve for A

$$\nabla\wedge\nabla\wedge A = J$$
$$-\nabla^2 A = J$$

$A = \mathcal{G}(J)$, the Newtonian potential of J, i.e.

$$A(x) = \frac{1}{4\pi}\int_{x'} \frac{J(x')}{|x-x'|}$$

Taylor expand the Green's function kernel:

$$\frac{1}{|x-x'|} = \frac{1}{|x|} + \frac{x\cdot x'}{|x|^3} + \cdots$$

$$A(x) = \frac{1}{4\pi}\left[\frac{1}{|x|}\int J + \frac{1}{|x|^3}\, x\cdot\int_{x'} x' J + \cdots\right]$$

Recall $\rho_t + \nabla\cdot J = 0$, $\nabla\cdot E = \rho$

magnetostatics $\Rightarrow$ E, J are constant

$\Rightarrow \rho$ is constant

$\Rightarrow \boxed{\nabla\cdot J = 0}$

Claim $\int J = 0$

Can write J as the divergence of a decaying 2nd order tensor:

$$\nabla\cdot(Jx) = \underbrace{(\nabla\cdot J)}_{0}x + \underbrace{J\cdot\nabla x}_{J,\ \text{since } \nabla x = \underline{\underline{I}}}$$

So $\int J = \int \nabla\cdot(Jx) = 0$

if J decays rapidly.

Claim $\int xJ = -\int Jx$

i.e. $\int(xJ + Jx) = 0$. Well,

$$\int\left(\nabla\cdot(Jxx) = \underbrace{(\nabla\cdot J)}_{0}xx + \underbrace{(J\cdot\nabla x)}_{J}x + x\underbrace{(J\cdot\nabla x)}_{J}\right)$$

i.e. $0 = \int(Jx + xJ)$

So
$$x\cdot\int' x' J = \tfrac{1}{2}x\cdot\int'(x'J + x'J)$$
$$= x\cdot\int'(x'J - Jx')\tfrac{1}{2}$$
$$= -x\wedge\int'\underbrace{(x'\wedge J)\left(\tfrac{1}{2}\right)}_{\text{Call } \mathcal{M}(x') = \text{magnetic moment density}}$$

So we define the magnetic moment density $\mathcal{M}$ by

$$\mathcal{M}(x') := \tfrac{1}{2}[x'\wedge J(x')]$$

and we define the magnetic moment m by

$$m := \int_{x'}\mathcal{M}(x')$$

This gives:

$$\boxed{A(x) = \frac{1}{4\pi}\frac{m\wedge x}{|x|^3}}$$ magnetic dipole vector potential

So

$$B = \nabla\wedge A$$
$$= \frac{1}{4\pi}\nabla\wedge\left(m\wedge\frac{x}{|x|^3}\right)$$
$$= \frac{1}{4\pi}\left[m\underbrace{\nabla\cdot\left(\frac{x}{|x|^3}\right)}_{4\pi\delta(x)} - m\cdot\nabla\left(\frac{x}{|x|^3}\right)\right]$$

$$\nabla\left(\frac{x}{|x|^3}\right) = \frac{(\nabla x)|x|^3 - (3|x|^2\hat{x})x}{|x|^6}$$
$$= \frac{\underline{\underline{I}} - 3\hat{x}\hat{x}}{|x|^3}$$

So far from the current source,

$$\boxed{B = \frac{1}{4\pi}\,\frac{3\hat{x}\hat{x}\cdot m - m}{|x|^3}}$$

Field of a magnetized object $\mathcal{r} = r - r'$

Recall the definition of magnetic moment:

$$m := \int_{r'} \tfrac{1}{2} r' \wedge J(r') \quad \Big| \quad \underset{\uparrow \text{ location of source}}{m(r)} = \int_{r'} \tfrac{-1}{2} \mathcal{r} \wedge J(r')$$

This gives the potential

$$A(r) = \frac{1}{4\pi} \frac{m \times \hat{r}}{|r|^3} \quad \Big| \quad A(r) = \frac{1}{4\pi} \frac{\overset{\text{location of source}}{m(r')} \times \hat{\mathcal{r}}}{|\mathcal{r}|^2}$$

Now suppose there is a bound current source dJ located at each infinitesimal region d^3r' of space.

This current produces a piece of magnetic moment:

$$dM(r') = M(r')\, d^3r'$$

↑ location of source.

$$A(r) = \frac{1}{4\pi} \int_{r'} \frac{M(r') \wedge \hat{\mathcal{r}}}{|\mathcal{r}|^2}$$

$$= \frac{1}{4\pi} \int_{r'} M \wedge \nabla\left(\frac{1}{\mathcal{r}}\right)$$

$$= \frac{1}{4\pi} \int_{r'} -\nabla \wedge \left(\frac{M}{\mathcal{r}}\right) + \frac{1}{\mathcal{r}}(\nabla \wedge M)$$

$$= \frac{1}{4\pi} \left[\oint \frac{M \wedge \hat{n}}{\mathcal{r}} + \int \frac{\nabla \wedge M}{\mathcal{r}} \right]$$

Want to argue that the net bound current is concentrated on the boundary (when the magnetic moment is presumed to drop discontinuously there).

Magnetic auxiliary field H

Assume magnetostatics.

Assume the net current J is composed of some free current J_f plus some bound current J_b that is divergenceless.

The potential caused by the net and free currents is:

$$A = \frac{1}{4\pi}\int \frac{J}{r}, \quad A_f = \frac{1}{4\pi}\int \frac{J_f}{r}$$

The potential caused by the bound charge is approximately the potential caused by its dipole moment per unit volume, since we assume that the bound currents come in localized units with net neutral charge:

$$A_b = \frac{1}{4\pi}\int \frac{J_b}{r}$$
$$\simeq \frac{1}{4\pi}\int M \wedge \nabla\left(\frac{1}{r}\right)$$
$$= \frac{1}{4\pi}\int \frac{\nabla \wedge M}{r}$$

So the net potential is:

$$A = \frac{1}{4\pi}\int \frac{J_f}{r} + \frac{\nabla\wedge M}{r}$$

But

$$A = \frac{1}{4\pi}\int \frac{J}{r}.$$

Since A is uniquely determined by J,

$$J = J_f + \nabla \wedge M.$$

i.e. $\boxed{J_b = \nabla \wedge M}$.

Recall Ampere's law:

$$\nabla \wedge B = J = J_f + \nabla \wedge M$$

So $\nabla \wedge \underbrace{(B - M)}_{\text{Call } H.} = J_f$

So $\boxed{H := B - M}$

Can we say that $H = B_f$?

No: $\nabla \cdot H = -\nabla \cdot M \neq 0$ in general.

[See Griffiths p261]

Linear Medium

Assume $M = \chi B$

$$H = B - M$$
$$= B - \chi B$$
$$= (1-\chi) B$$

So $B = (1-\chi)^{-1} H$

So $M = \chi (1-\chi)^{-1} B$

Define $\chi_m := \chi(1-\chi)^{-1} \Rightarrow \chi_m - \chi_m \chi = \chi$
$\Rightarrow \chi_m = (\chi_m + 1)\chi$

(So $\chi = (\chi_m + 1)^{-1}\chi_m$)

$$\boxed{M = \chi_m H}$$

$$B = H + M$$
$$= (1 + \chi_m) H$$

Define $\boxed{\mu := 1 + \chi_m}$

So $\boxed{B = \mu H}$