# Derivation of Navier-Stokes

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## 1 Derivation of Conservation Laws

### 1.1 Context and Conventions

By default quantities are functions of space $\mathbf{x}$ and time $t$.
Let $\mathbf{u}$ be the velocity field (which is convecting the continuum).
Let $\alpha$, $\beta$, and $\mathbf{q}$ stand for arbitrary (convected) quantities.
Let $U(t)$ stand for an arbitrary convected region (volume element). ($U(t)$ is simply connected with smooth boundary.)
Let $\partial U$ denote the boundary of the region $U$.
Let $\int := \int_{U(t)}$, and let $\oint := \int_{\partial U(t)}$, i.e. the default domain of integration is the arbitrary convected volume element.
Let $\mathbf{n}$ denote the outward unit normal to $\partial U$.

### 1.2 Kinetics Calculus

**Definitions.**
Let $d_t := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ denote the **convective derivative**.
Let $\eth_t := \alpha \mapsto (\frac{\partial}{\partial t}\alpha + \nabla \cdot (\mathbf{u}\alpha))$ denote the **conservative derivative**. (I made up this term and this symbol. $\eth_t$ is supposed to be reminiscent of the averaging operator $⨍$ and $\delta$ signifying differentiation.)

**Leibnitz rules.**
Observe that $\eth_t\alpha = d_t\alpha + (\nabla \cdot \mathbf{u})\alpha$. Hence:
$d_t(\alpha\beta) = (d_t\alpha)\beta + \alpha(d_t\beta)$.
$\eth_t(\alpha\beta) = d_t(\alpha\beta) + (\nabla \cdot \mathbf{u})\alpha\beta$
$= (d_t\alpha)\beta + \alpha(d_t\beta) + (\nabla \cdot \mathbf{u})\alpha\beta$
$= (\eth_t\alpha)\beta + \alpha(d_t\beta)$
$= (d_t\alpha)\beta + \alpha(\eth_t\beta)$.

**Gauss's Theorem**
$\int \nabla\alpha = \oint \mathbf{n}\alpha$, $\boxed{\int \nabla \cdot \mathbf{q} = \oint \mathbf{n} \cdot \mathbf{q}}$ , and $\int \nabla \times \mathbf{q} = \oint \mathbf{n} \times \mathbf{q}$.

**Reynolds' Transport Theorem.**

$\boxed{\frac{d}{dt}\int \alpha = \int \eth_t\alpha}$ , i.e.
$\frac{d}{dt}\int_{U(t)} \alpha = \int_{U(t)}(\frac{\partial}{\partial t}\alpha + \nabla \cdot (\mathbf{u}\alpha))$

Justification. (Convection applies to $U(t)$, not $\alpha(\mathbf{x}, t)$.) Use time-splitting on the time increment: alternatively allow $\alpha$ and $U(t)$ to evolve. Then apply Gauss's Theorem.
$\frac{d}{dt}\int_{U(t)} \alpha = \int \frac{\partial}{\partial t}\alpha + \oint \mathbf{n} \cdot \mathbf{u}\alpha = \int \frac{\partial}{\partial t}\alpha + \int \nabla \cdot (\mathbf{u}\alpha))$

### 1.3 Conservation Laws

#### 1.3.1 Definitions of Quanitities

Let $\rho$ denote mass per volume.
Observe that $\mathbf{u}$ is momentum per mass.
Let $e$ denote internal energy per volume.
Observe that $\frac{1}{2}\rho u^2$ is macroscopic kinetic energy per volume.
Let $\mathbf{f}$ denote body force (force per unit mass).
Let $\sigma$ denote the stress tensor: $\mathbf{n} \cdot \sigma$ is the surface force per unit area on an infinitesimal surface element orthogonal to $\mathbf{n}$, where $\mathbf{n}$ points away from the side of the interface on which the force acts. Thus $\sigma_{ij} := \mathbf{e}_i \cdot \sigma \cdot \mathbf{e}_j$ is the component in the direction $\mathbf{e}_j$ of the surface force acting on the low side of an infinitesimal surface orthogonal to $\mathbf{e}_i$. This stress tensor representation of surface forces is justified by noting that the sum of the forces must be zero on an infinitesimal tetrahedron with 3 sides aligned with the principle axes. Application of conservation of angular momentum to an infinitesimal cube aligned with the principle axes shows that the stress tensor is symmetric. [cite Rutherford Aris.]
Let $\mathbf{q}$ denote the heat flux: $\mathbf{q} \cdot \mathbf{n}$ is the rate of external flow of heat per unit area across an infinitesimal surface element orthogonal to $\mathbf{n}$ (i.e. the component of the flow of heat in the direction of $\mathbf{n}$).

#### 1.3.2 Conservation of Mass

$\frac{d}{dt}\int \rho = 0$, i.e. $\boxed{\eth_t\rho = 0}$, i.e. $\boxed{\rho_t + \nabla \cdot \rho\mathbf{u} = 0}$.

#### 1.3.3 Conservation of Momentum

$\frac{d}{dt}\int \rho\mathbf{u} = \oint \mathbf{n} \cdot \sigma + \int \rho\mathbf{f}$, i.e.

$\boxed{\eth_t(\rho\mathbf{u}) = \nabla \cdot \sigma + \rho\mathbf{f}}$ (conservation form).

Simplify using Leibnitz rule and conservation of mass: $\eth_t(\rho\mathbf{u}) = \cancel{(\eth_t\rho)}\mathbf{u} + \rho(d_t\mathbf{u})$. So:
$\boxed{\rho d_t\mathbf{u} = \nabla \cdot \sigma + \rho\mathbf{f}}$ (simplified form).

#### 1.3.4 Conservation of Energy

$\frac{d}{dt}\int(\rho e + \frac{1}{2}\rho\mathbf{u} \cdot \mathbf{u}) = \oint \mathbf{n} \cdot \sigma \cdot \mathbf{u} + \int \mathbf{u} \cdot \rho\mathbf{f} - \oint \mathbf{n} \cdot \mathbf{q}$, i.e.
$\boxed{\eth_t(\rho e + \frac{1}{2}\rho\mathbf{u} \cdot \mathbf{u}) = \nabla \cdot (\sigma \cdot \mathbf{u}) + \mathbf{u} \cdot \rho\mathbf{f} - \nabla \cdot \mathbf{q}}$
(conservation form)

We can simplify using Leibnitz and the previous conservation laws.

Simplify the following terms using Leibnitz rules and conservation of mass:
$\eth_t(\rho e) = \rho(d_t e) + \cancelto{0}{(\eth_t\rho)}e$.
$\eth_t(\frac{1}{2}\rho\mathbf{u} \cdot \mathbf{u}) = \rho d_t(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) = \rho(d_t\mathbf{u}) \cdot \mathbf{u}$.
$\nabla \cdot (\sigma \cdot \mathbf{u}) = \frac{\partial}{\partial x_i}(\sigma_{ij}\mathbf{u}_j) = \sigma_{ij}\frac{\partial}{\partial x_i}\mathbf{u}_j + (\frac{\partial}{\partial x_i}\sigma_{ij})\mathbf{u}_j$
$= \sigma \cdot\cdot \nabla\mathbf{u} + (\nabla \cdot \sigma) \cdot \mathbf{u}$ (where $\cdot\cdot$ here denotes contraction of corresponding indices).

Now put the terms together and invoke the simplified form of the conservation of momentum equation. Get:
$\rho d_t e + (\rho d_t\mathbf{u}) \cdot \mathbf{u} = \sigma \cdot\cdot \nabla\mathbf{u} + \underbrace{(\nabla \cdot \sigma) \cdot \mathbf{u} + \rho\mathbf{f} \cdot \mathbf{u}}_{(\rho d_t\mathbf{u}) \cdot \mathbf{u}} - \nabla \cdot \mathbf{q}$.

So $\boxed{\rho d_t e = \sigma \cdot\cdot \nabla\mathbf{u} - \nabla \cdot \mathbf{q}}$ (simplified form)

# 2 Derivation of Navier-Stokes

## 2.1 Constitutive Relations

### 2.1.1 Stress-Strain (terse)

Assume that $\sigma = -pI + \tau$ where

| | | |
|---|---|---|
| $p$ | = | pressure |
| $I$ | = | identity tensor (2nd order) |
| $\tau$ | = | viscous/shear stress tensor |

The viscous stress tensor is assumed to depend linearly on the rate-of-strain tensor $\nabla\mathbf{u}$. This tensor is the sum of its symmetric and antisymmetric parts. Constant antisymmetric rate-of-strain tensors correspond bijectively with rigid-body rotations. The viscous stress tensor is assumed to be zero for a fluid undergoing rigid-body rotation. Then the viscous stress tensor $\tau$ must be a linear function of the even part of the rate-of-deformation tensor, $D := \frac{1}{2}((\nabla\mathbf{u})^T + \nabla\mathbf{u})$.

So $\tau_{ij} = K_{ijkl}D_{kl}$ for some fourth-order tensor $K$. Assume that $K$ is isotropic. Then $K_{ijkl}$ is a linear combination of products of $\delta$'s:

$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \tilde{\mu}\delta_{ik}\delta_{jl} + \tilde{\nu}\delta_{il}\delta_{jk}$.

Since we know that $D_{kl}$ is symmetric, we write

$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$.

So $\tau_{ij} = \lambda\delta_{ij}D_{kk} + 2\mu D_{ij}$

So $\nabla\cdot\tau = \lambda\nabla\nabla\cdot\mathbf{u} + \mu(\nabla\nabla\cdot\mathbf{u} + \nabla\cdot\nabla\mathbf{u})$

So $\boxed{\rho d_t\mathbf{u} = \lambda\nabla\nabla\cdot\mathbf{u} + \mu(\nabla\nabla\cdot\mathbf{u} + \nabla\cdot\nabla\mathbf{u}) - \nabla p + \rho\mathbf{f}}$

Assume that the fluid is incompressible: $\boxed{\nabla\cdot\mathbf{u} = 0}$

Then $\nabla\cdot\tau = \mu\triangle\mathbf{u}$.

So $\boxed{\rho d_t\mathbf{u} = \mu\triangle\mathbf{u} - \nabla p + \rho\mathbf{f}}$