Taylor's Theorem

1-D: $f(x_0+\Delta x) = \sum_{k=0}^{r-1} \frac{1}{k!}\Delta x^k \partial_x^k f(x_0)$

$\qquad + \frac{1}{r!}\Delta x^r \partial_x^r f(x_0+t\Delta x)$

where $0 \le t \le 1$. [Can t depend continuously on Δx and f? differentiably?]

(†) $\mathbb{R}^n$: $f(\underline{x}_0+\underline{\Delta x}) = \sum_{s=0}^{r-1}\frac{1}{s!}\sum_{\underline{i}\in \mathbb{Z}_n^s} \Delta x_{i_1}\cdots\Delta x_{i_s}\partial_{i_1}\cdots\partial_{i_s} f(\underline{x}_0)$

$\qquad + \frac{1}{r!}\sum_{\underline{i}\in\mathbb{Z}_n^r}\Delta x_{i_1}\cdots\Delta x_{i_r}\partial_{i_1}\cdots\partial_{i_r} f(\underline{x}_0+t\underline{\Delta x})$

where $0 \le t \le 1$ $\quad (\partial_\ell \equiv \partial_{x_\ell})$

Pf. Let $F(t) = f(\underline{x}_0 + t\underline{\Delta x})$

Apply the chain rule and the 1-D Taylor thm:

$F'(t) = \sum_i \Delta x_i \partial_{x_i} f(\underline{x}_0 + t\underline{\Delta x})$

$F^{(s)}(t) = \sum_{\underline{i}\in\mathbb{Z}_n^s}\Delta x_{i_1}\cdots\Delta x_{i_s}\partial_{x_{i_1}}\cdots\partial_{x_{i_s}} f(\underline{x}_0+t\underline{\Delta x})$

etc.

(*) $\mathbb{R}^n$ (more popular version)

$f(\underline{x}_0+\underline{\Delta x}) = \sum_{s=0}^{r-1}\sum_{\underline{k}\in I_n^s}\frac{1}{\underline{k}!}\underline{\Delta x}^{\underline{k}}\partial_{\underline{x}}^{\underline{k}} f(\underline{x}_0)$

$\qquad + \sum_{\underline{k}\in I_n^r}\frac{1}{\underline{k}!}\underline{\Delta x}^{\underline{k}}\partial_{\underline{x}}^{\underline{k}} f(\underline{x}_0+t\underline{\Delta x})$ where $0\le t\le 1$

where:

$I_n^s \equiv \{\underline{k}: \mathbb{Z}_n \to \mathbb{Z}^+ \mid \sum_{i=1}^n k_n = s\}$

where $\mathbb{Z}^+$ = nonnegative integers

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

$\underline{k}! \equiv \prod_{i=1}^n k_i! = k_1!\cdots k_n!$

$\underline{\Delta x}^{\underline{k}} \equiv \Delta x_1^{k_1}\cdots\Delta x_n^{k_n}$

$\partial_{\underline{x}}^{\underline{k}} \equiv \partial_{x_1}^{k_1}\cdots\partial_{x_n}^{k_n}$

example: 2-D

$f(a_1+\Delta x_1, a_2+\Delta x_2)$

$= \sum_{s=0}^{r-1}\sum_{\substack{(k_1,k_2):\\ k_1+k_2=s}}\frac{1}{k_1!}\frac{1}{k_2!}\Delta x_1^{k_1}\Delta x_2^{k_2}\partial_{x_1}^{k_1}\partial_{x_2}^{k_2} f(\underline{a})$

$\qquad + \sum_{\substack{(k_1,k_2):\\ k_1+k_2=r}}$ [same stuff] $f(\underline{a}+t\underline{\Delta x})$

Pf Need to see that the s^{th} term is the same for both Taylor sums.

To see this, apply $\partial_{x_1}^{\ell_1}\cdots\partial_{x_n}^{\ell_n}$ to the sum and observe that you get $\partial_{x_1}^{\ell_1}\cdots\partial_{x_n}^{\ell_n} f(\underline{a})$ at $\underline{a}$.

Suffices to show that each term is correct (i.e. has correct coefficient).

Note:

One can also verify the first Taylor series this way: Apply $\partial_{j_1}\cdots\partial_{j_s}$ & eval. at $\underline{\Delta x}=0$. Observe that the only terms which survive are those for which a permutation of the sequence of partials $\partial_{j_1}\cdots\partial_{j_s}$ can be matched up to the $\Delta x_{i_1}\cdots\Delta x_{i_s}$. These all have the same coefficient $\partial_{j_1}\cdots\partial_{j_s} f(\underline{x}_0+t\underline{\Delta x})$, since the partials can also be matched up, and partials commute. So it suffices to show that:

$\partial_{j_1}\cdots\partial_{j_s}\sum_{\underline{i}\in\mathbb{Z}_n^s}\Delta x_{i_1}\cdots\Delta x_{i_s} = s!$

But LHS $= \sum_{\underline{i}\in\mathbb{Z}_n^s}\partial_{j_1}\cdots\partial_{j_s}\Delta x_{i_1}\cdots\Delta x_{i_s}$

$= \sum_{\underline{i}\in\mathbb{Z}_n^s}\sum_{\underline{k}\in P_s}(\partial_{j_{k_1}}\Delta x_{i_1})\cdots(\partial_{j_{k_s}}\Delta x_{i_s})$

where P_s = set of all permutations of $\mathbb{Z}_s$. (Comes from repeated application of product rule.)

(summing over all ways of matching differential operators to a particular choice of Δx's.)

$= \sum_{\underline{i}\in\mathbb{Z}_n^s}\sum_{\underline{k}\in P_s}\delta_{j_{k_1}-i_1}\cdots\delta_{j_{k_s}-i_s}$.

where $\delta_\ell = \begin{cases}1 & \text{if } \ell=0\\ 0 & \text{else}\end{cases}$

$= \sum_{\underline{k}\in P_s} 1$ (There is only one $\underline{i}$ that gives nonzero for a given $\underline{k}$)

$= s!$, as desired.

Another argument for equivalence of (†) and (*):

Pick arbitrary term s, $\underline{k}$ from (*). The number of times this term appears in (†) is $\binom{s}{\underline{k}}$ = s-choose-$\underline{k}$ = $\frac{s!}{\underline{k}!}$

(i.e. the number of ways of choosing the s indices $i_1\cdots i_s$ so that Δx_i appears k_i times for each i).

Vector notation for (†):

(†) $f(\underline{x}_0+\underline{\Delta x}) = \sum_{s=0}^{r-1}\frac{1}{s!}(\underline{\Delta x}\cdot\nabla)^s f(\underline{x}_0)$

$\qquad + \frac{1}{r!}(\Delta x\cdot\nabla)^r f(\underline{x}_0+t\underline{\Delta x})$

where $0\le t\le 1$.

Operator notation for (†):

(†) $f(\underline{x}_0+\underline{\Delta x}) = \sum_{s=0}^{r-1}\frac{1}{s!}\left(\sum_{i=1}^n \Delta x_i\partial_i\right)^s f(\underline{x}_0)$

$\qquad + \frac{1}{r!}\left(\sum_{i=1}^n\Delta x_i\partial_i\right)^r f(\underline{x}_0+t\underline{\Delta x})$

where $0\le t\le 1$

Proof of 1-D Taylor:

WLOG $x_0 = 0$.

$$f(h) = f(0) + \boxed{\int_0^h f'(x)\,dx} \leftarrow R_0(h)$$

$$\therefore R_0(h) = \int_0^h [(x-h)f']' - (x-h)f''\,dx$$

$$= \underbrace{[(h-x)f']_h^0}_{hf'(0)} + \underbrace{\int_0^h (h-x)f''\,dx}_{R_1(h)}$$

$$R_1(h) = \int_0^h \left[\frac{(h-x)^2}{-2}f''\right]' - \frac{(h-x)^2}{-2}f'''\,dx$$

$$= \underbrace{\left[\frac{(h-x)^2}{2}f''\right]_h^0}_{\frac{1}{2}h^2 f''(0)} + \underbrace{\int_0^h \frac{1}{2}(h-x)^2 f'''\,dx}_{R_2(h)}$$

Claim

$$R_n(h) = \int_0^h \frac{1}{n!}(h-x)^n f^{(n+1)}\,dx$$

Inductive verification:

$$R_n(h) = \int_0^h \left[\frac{-1}{(n+1)!}(h-x)^{n+1} f^{(n+1)}\right]' + \frac{(h-x)^{n+1}}{(n+1)!} f^{(n+2)}\,dx$$

$$= \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(0) + R_{n+1}(h) \quad ✓$$

Sampling form of remainder

$$R_n(h) = \int_0^h \frac{1}{n!}(h-x)^n f^{(n+1)}(x)\,dx$$

[Recall weighted mean value theorem:
Suppose $g \geq 0$ on Ω,
and f & g cont. on Ω.
Then $\int_\Omega fg = f(p)\int_\Omega g$
for some $p \in \Omega$.]

So $R_n(h) = f^{(n+1)}(p)\int_0^h \frac{1}{n!}(h-x)^n\,dx$

for some $p \in [0,h]$

So $R_n(h) = f^{(n+1)}(p)\dfrac{h^{n+1}}{(n+1)!}$ ✓

$\boxed{\text{Can } p \text{ depend continuously on } h \text{ and } f\text{?}}$

1-D Taylor

$$f(h) = f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \cdots + \frac{1}{n!}h^n f^{(n)}(0) + R_n(h)$$

where

$$R_n(h) = \int_0^h \frac{1}{n!}(h-x)^n f^{(n+1)}(x)\,dx$$

$$= \frac{1}{(n+1)!}h^{n+1} f^{(n+1)}(p)$$

where $p \in [0,h]$.

COR

$$f(x_0+\Delta x) = \left[\sum_{k=0}^{n} \frac{1}{k!}\Delta x^k f^{(k)}(x_0)\right] + R_n(\Delta x)$$

where

$$R_n(\Delta x) = \int_0^{\Delta x} \frac{1}{n!}(\Delta x - \widetilde{\Delta x})^n f^{(n+1)}(x_0 + \widetilde{\Delta x})\,d\widetilde{\Delta x}$$

$$= \frac{1}{(n+1)!}\Delta x^{n+1} f^{(n+1)}(x_0 + p)$$

for some $p \in [0, \Delta x]$.

Mean Value Theorem (see Courant, Differential & Integral Calculus, Volume 1.)

Let Ω be a connected, compact set.
Let $f: \Omega \to \mathbb{R}$ be continuous
Let $g: \Omega \to \mathbb{R}$, $g \geq 0$, be continuous (integrable enough?)
Then $\int_\Omega fg = f(\xi)\cdot\int_\Omega g$ for some $\xi \in \Omega$.

Pf Let $m = \min_\Omega f$.
Let $M = \max_\Omega f$.
Since $g \geq 0$, $m\,g(x) \leq f(x)g(x) \leq Mg(x)$ $(\forall x \in \Omega)$
So $m\int_\Omega g(x)\,dx \leq \int_\Omega f(x)g(x)\,dx \leq M\int_\Omega g(x)\,dx$.
So $\int f(x)g(x)\,dx = \mu\int_\Omega g(x)$, where $m \leq \mu \leq M$.
[$f(\Omega)$ is connected and compact (since f is continuous).
So f attains its minimum & maximum on Ω.
Let $f(x_1) = m$ and $f(x_2) = M$.
Let $\tau: [0,1] \to \Omega$ be a continuous path from x_1 to x_2. (So $\tau(x_1) = 0$ and $\tau(x_2) = 1$.)
Note $f \circ \tau: [0,1] \to [m,M]$ is continuous & so onto.
So $\exists t \in [0,1]$ s.t. $f \circ \tau(t) = \mu$.
So $\exists \xi = \tau(t) \in \Omega$ s.t. $f(\xi) = \mu$.
So $\int_\Omega fg = f(\xi)\int_\Omega g$, as desired.

$\boxed{\text{Can } \xi \text{ depend continuously on } \Omega, f, \text{ and } g\text{?}}$

Taylor's Theorem (For sharp proofs)

Let $f(\underline{x}) \in \mathcal{C}^r$, $\underline{x} \in \mathbb{R}^n$. Write:

$$f(\underline{x}_0 + \underline{\Delta x}) = \sum_{s=0}^{r} \frac{1}{s!} \sum_{\underline{i} \in \mathbb{Z}_n^s} \Delta x_{i_1} \cdots \Delta x_{i_s} \partial_{i_1} \cdots \partial_{i_s} f(\underline{x}_0)$$
$$+ \text{error}(\underline{\Delta x})$$

Then: $\lim\limits_{|\underline{\Delta x}| \to 0} \dfrac{\text{error}(\underline{\Delta x})}{|\underline{\Delta x}|^r} = 0$

i.e. $\text{error}(\underline{\Delta x}) = o(|\underline{\Delta x}|^r)$

Proof

Let $\tilde{f}_s(\underline{x}_0, \underline{\Delta x}) = \sum_{\underline{i} \in \mathbb{Z}_n^s} \Delta x_{i_1} \cdots \Delta x_{i_s} \partial_{i_1} \cdots \partial_{i_s} f(\underline{x}_0)$

Let $\tilde{f}(\underline{x}_0, \underline{\Delta x}) = \sum_{s=0}^{r} \frac{1}{s!} \tilde{f}_s(\underline{x}_0, \underline{\Delta x})$

So $\text{error}(\underline{\Delta x}) = f(\underline{x}_0 + \underline{\Delta x}) - \tilde{f}(\underline{x}_0, \underline{\Delta x})$

Use Taylor's theorem to write:

$$f(\underline{x}_0 + \underline{\Delta x}) = \sum_{s=0}^{r-1} \frac{1}{s!} \tilde{f}_s(\underline{x}_0, \underline{\Delta x}) + \frac{1}{r!} \tilde{f}_r(\underline{\xi}, \underline{\Delta x})$$

where $\underline{\xi} \in [\underline{x}_0, \underline{x}_0 + \underline{\Delta x}]$.

$$f(\underline{x}_0 + \underline{\Delta x}) = \sum_{s=0}^{r} \frac{1}{s!} \tilde{f}_s(\underline{x}_0, \underline{\Delta x}) + \underbrace{\frac{1}{r!}\left(\tilde{f}_r(\underline{\xi}, \underline{\Delta x}) - \tilde{f}_r(\underline{x}_0, \underline{\Delta x})\right)}_{\text{error}(\underline{\Delta x}).}$$

$$\text{So } \text{error}(\underline{\Delta x}) = \frac{1}{r!} \sum_{\underline{i} \in \mathbb{Z}_n^r} \Delta x_{i_1} \cdots \Delta x_{i_r} \underbrace{\left[\partial_{i_1} \cdots \partial_{i_r} f(\underline{\xi}) - \partial_{i_1} \cdots \partial_{i_r} f(\underline{x}_0)\right]}_{\text{Call } A_{i_1 \cdots i_r}(\underline{\Delta x}).}$$

[Since $\text{error}(\underline{\Delta x}) = f(\underline{x}_0 + \underline{\Delta x}) - \sum_{s=0}^{r} \frac{1}{s!} \tilde{f}_s(\underline{x}_0, \underline{\Delta x})$, error is continuous on the compact set $[0, \underline{\Delta x}]$, so it is bounded.]

$A_{i_1 \cdots i_r}(\underline{\Delta x})$ is bounded in any compact domain $|\underline{\Delta x}| < \delta$, because $f \in \mathcal{C}^r$.

Also, $A_{i_1 \cdots i_r}(\underline{\Delta x}) \to 0$ as $\underline{\Delta x} \to 0$, since $\partial_{i_1} \cdots \partial_{i_r} f(\xi) \to \partial_{i_1} \cdots \partial_{i_r} f(\underline{x}_0)$ as $\underline{\Delta x} \to 0$

So $\dfrac{\text{error}(\underline{\Delta x})}{|\underline{\Delta x}|^r} = \dfrac{1}{r!} \sum_{\underline{i} \in \mathbb{Z}^r} \hat{\Delta x}_{i_1} \cdots \hat{\Delta x}_{i_r} A_{i_1 \cdots i_r}(\underline{\Delta x})$

(where $\hat{\Delta x}_i = \dfrac{\Delta x_i}{|\underline{\Delta x}|}$)

So $\left|\dfrac{\text{error}(\underline{\Delta x})}{|\underline{\Delta x}|^r}\right| \leq \dfrac{1}{r!} \sum_{\underline{i} \in \mathbb{Z}^r} A_{i_1 \cdots i_r}(\underline{\Delta x})$

$\leq \dfrac{1}{r!} \sum n^r \max |A_{i_1 \cdots i_r}(\underline{\Delta x})|$

$\to 0$ as $\underline{\Delta x} \to 0$.

Special cases

$\boxed{\mathcal{C}^1}$ Let $f \in \mathcal{C}^1$.

Then $f(\underline{x}_0 + \underline{\Delta x}) = f(\underline{x}_0) + \Delta x_i \partial_i f(\underline{x}_0) + \text{error}(\underline{\Delta x})$

where $\lim\limits_{|\Delta x| \to 0} \dfrac{\text{error}(\underline{\Delta x})}{|\underline{\Delta x}|} = 0$

i.e. $\text{error}(\underline{\Delta x}) = o(|\Delta x|)$

$\boxed{\mathcal{C}^2}$ Let $f \in \mathcal{C}^2$

Then $f(\underline{x}_0 + \underline{\Delta x}) = f(\underline{x}_0) + \Delta x_i \partial_i f(\underline{x}_0) + \Delta x_i \Delta x_j \partial_i \partial_j f(\underline{x}_0) + \text{error}(\underline{\Delta x})$

where $\lim\limits_{|\Delta x| \to 0} \dfrac{\text{error}(\underline{\Delta x})}{|\underline{\Delta x}|^2} = 0$

i.e. $\text{error}(\underline{\Delta x}) = o(|\Delta x|^2)$