Vector Calculus

Gauss's Law (FTC)

$* \int_{\partial V} n_i f \, dA = \int_V \frac{\partial f}{\partial x_i} dV$

Thus: $\int_{\partial V} n_j u_j \, dA = \int_V \partial_j u_j \, dV$

$\left(\int_{\partial V} n \cdot u \, dA = \int_V \nabla \cdot u \, dV \right)$

$\int_{\partial V} n_i u_j \, dA = \int_V \partial_i u_j \, dV$

$\left(\int_{\partial V} n u \, dA = \int_V \nabla u \, dV \right)$

$\int_{\partial V} \varepsilon_{ijk} n_j u_k \, dA = \int_V \varepsilon_{ijk} \partial_j u_k \, dV$

$\left(\int_{\partial V} n \times u \, dA = \int_V \nabla \times u \, dV \right)$

Stoke's Theorem (FTC)

$\oint_{\partial A} u \cdot dl = \int_A (\nabla \times u) \cdot n \, dA$

i.e. $\oint_{\partial A} u_i \, dl_i = \int_A \varepsilon_{ijk} \partial_j u_k n_i \, dA$

(can derive Stoke's theorem by dividing up the surface into small surface elements that are approximately linear, and applying Gauss's law to a "sheets" of infinitesimal thickness that contains each surface element.)

Green's Identities (Integration by parts)

$\nabla \cdot (\psi (c \nabla \varphi)) = c \nabla \psi \cdot \nabla \varphi + \psi \nabla \cdot (c \nabla \varphi)$

i.e.: $\partial_{x_j} (\psi (c \partial_{x_j} \varphi)) = c (\partial_{x_j} \psi)(\partial_{x_j} \varphi) + \psi \partial_{x_j} (c \partial_{x_j} \varphi)$

So: $\int_{\partial \Omega} \psi c (\hat{n} \cdot \nabla \varphi) dS = \int_{\Omega} c \nabla \psi \cdot \nabla \varphi + \psi \nabla \cdot (c \nabla \varphi) d\Omega$

Note $\hat{n} \cdot \nabla \varphi \equiv \frac{\partial \varphi}{\partial \hat{n}}$

When $c = 1$ this becomes statement about Laplacian:

$\nabla \cdot (\psi \nabla \varphi) = \nabla \psi \cdot \nabla \varphi + \psi \nabla^2 \varphi$

So $\int_{\partial \Omega} \psi \frac{\partial}{\partial n} \varphi = \int_{\Omega} \nabla \psi \cdot \nabla \varphi \, d\Omega + \int_{\Omega} \psi \nabla^2 \varphi \, d\Omega$

The basic idea is that in an integrand you can transfer a differential operator from one multiplicand in a product to the other and the only difference is something you have to integrate on the boundary. So there is a "Green's identity" for every differential operator (applied to a product of functions).

example:

$\nabla \wedge (u \underline{A}) = (\nabla u) \wedge A + u (\nabla \wedge A)$

So $\oint_{\partial \Omega} n \wedge (u A) = \int_{\Omega} (\nabla u) \wedge A + \int_{\Omega} u (\nabla \wedge A)$

Product & Composition rules

$* \ \underline{\nabla}(uv) = (\nabla u) v + u (\nabla v)$

$\partial_i (uv) = (\partial_i u) v + u (\partial_i v)$

$* \ \underline{\nabla} \cdot (u \underline{A}) = (\underline{\nabla} u) \cdot \underline{A} + u (\underline{\nabla} \cdot \underline{A})$

$(\partial_i (u A_i) = (\partial_i u) A_i + u (\partial_i A_i))$

$* \ \underline{\nabla} \wedge (u \underline{A}) = (\underline{\nabla} u) \wedge \underline{A} + u (\nabla \wedge A)$

$(\varepsilon_{ijk} \partial_j (u A_k) = \varepsilon_{ijk} (\partial_j u) A_k + u \varepsilon_{ijk} \partial_j A_k)$

$* \ \underline{\nabla} \cdot (\underline{A} \wedge \underline{B}) = \underline{B} \cdot (\underline{\nabla} \wedge \underline{A}) - \underline{A} \cdot (\underline{\nabla} \wedge \underline{B})$

$\partial_i (\varepsilon_{ijk} A_j B_k) = B_k \varepsilon_{kij} \partial_i A_j - A_j \varepsilon_{jik} \partial_i B_k$

$* \ \nabla \wedge (\underline{A} \wedge \underline{B}) = (\underline{B} \cdot \nabla) A - B (\nabla \cdot A) - (A \cdot \nabla) B + A (\nabla \cdot B)$

$$\begin{aligned} \varepsilon_{ijk} \partial_j (\varepsilon_{klm} A_l B_m) &= \varepsilon_{ijk} \varepsilon_{lmk} \partial_j A_l B_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) (\partial_j A_l B_m) \\ &= \partial_j (A_i B_j - A_j B_i) \\ &= B_j \partial_j A_i - A_j \partial_j B_i + A_i \partial_j B_j - B_i \partial_j A_j \end{aligned}$$

$* \ \underline{\nabla} (\underline{A} \cdot \underline{B}) = (\underline{B} \cdot \underline{\nabla}) \underline{A} + (\underline{A} \cdot \underline{\nabla}) \underline{B} + \underline{B} \wedge (\underline{\nabla} \wedge \underline{A}) + \underline{A} \wedge (\underline{\nabla} \wedge \underline{B})$

$* \ \underline{\nabla} \wedge (\underline{\nabla} u) = \underline{0}$ curl ∘ grad $= \underline{0}$

$* \ \nabla \cdot (\nabla \wedge \underline{A}) = 0$ div ∘ curl $= \underline{0}$

$* \ \nabla \wedge (\nabla \wedge A) = \nabla (\nabla \cdot A) - \nabla^2 A$

Repeat

$* \ A \wedge (B \wedge C) = B (A \cdot C) - C (A \cdot B)$

- linear combination of B and C, so A is never free.
- plus sign on the term where the middle vector (B) is free
- minus sign on the term where the outside vector (C) is free.

$* \ \nabla \wedge (A \wedge B) = (B \cdot \nabla) A - (A \cdot \nabla) B + (\nabla \cdot B) A - (\nabla \cdot A) B$

- again, the minus sign appears when contracting with the inside term.

$* \ \nabla \wedge (\nabla \wedge A) = \nabla (\nabla \cdot A) - \nabla \cdot \nabla A$

- again, minus sign when contracting with the inside term.

$* \ \nabla \wedge (u \underline{A}) = (\nabla u) \wedge A + u (\nabla \wedge A)$

- list the scalar first.

Vector Calculus

$$a\cdot b = a_i b_i$$
$$[a\wedge b]_i = \varepsilon_{ijk} a_j b_k$$
$$[\nabla]_i = \partial_i$$
$$[\mathrm{grad}(f)]_i = [\nabla f]_i = \partial_i f$$
$$[\mathrm{div}(f)]_i = [\nabla\cdot f]_i = \partial_i f_i$$
$$[\mathrm{curl}(f)]_i = [\nabla\wedge f]_i = \varepsilon_{ijk}\partial_j f_k$$

alternating tensor
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i\,j\,k) \text{ is an even permutation of } (1\,2\,3) \\ -1 & \text{if } (i\,j\,k) \text{ is an odd permutation of } (1\,2\,3) \\ 0 & \text{if } (i\,j\,k) \text{ is not a permutation of } (1\,2\,3) \text{ (i.e. two of } i,j,\&\,k \text{ are equal)} \end{cases}$$

Kronecker delta
$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

$$\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{kij}\varepsilon_{klm} = \begin{cases} 1 & \text{if } =\varepsilon_{ijk}\varepsilon_{kij} \\ -1 & \text{if } = \varepsilon_{ijk}\varepsilon_{kji} \\ 0 & \text{otherwise} \end{cases} \quad \text{and no two of } i,j,k \text{ are equal.}$$

$$= \begin{cases} 1 & \text{if } i=l \,\&\, j=m \\ -1 & \text{if } i=m \,\&\, j=l \\ 0 & \text{otherwise} \end{cases} \quad \text{and no two of } i,j,k \text{ are equal}$$

$$= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

grad : $(\mathbb{R}^3\to\mathbb{R}) \to (\mathbb{R}^3\to\mathbb{R}^3)$
div : $(\mathbb{R}^3\to\mathbb{R}^3) \to (\mathbb{R}^3\to\mathbb{R})$
curl : $(\mathbb{R}^3\to\mathbb{R}^3) \to (\mathbb{R}^3\to\mathbb{R}^3)$

(So 5 possible ways to combine these.)
Combining the curl with a noncurl gives zero.
There are three combination operators:

Laplacian (= div grad) : $(\mathbb{R}^3\to\mathbb{R}) \to (\mathbb{R}^3\to\mathbb{R})$
grad div : $(\mathbb{R}^3\to\mathbb{R}^3) \to (\mathbb{R}^3\to\mathbb{R}^3)$
curl curl (= grad div − Laplacian) : $(\mathbb{R}^3\to\mathbb{R}^3) \to (\mathbb{R}^3\to\mathbb{R}^3)$

(4th) $u\cdot\nabla$: $(\mathbb{R}^3\to\mathbb{R}) \to (\mathbb{R}^3\to\mathbb{R})$
$(\mathbb{R}^3\to\mathbb{R}^3) \to (\mathbb{R}^3\to\mathbb{R}^3)$

Combinations of grad, div, & curl

- $\boxed{\text{div grad} = \text{Laplacian}}$ $(\nabla\cdot\nabla = \nabla^2)$
$$\nabla\cdot(\nabla f) = \partial_i\partial_i f = \partial_i^2 f = \nabla^2 f = \Delta f$$

- $\boxed{\text{curl grad} = \vec{0}}$
$$[\nabla\wedge(\nabla f)]_i = \varepsilon_{ijk}\partial_j(\nabla f)_k = \varepsilon_{ijk}\partial_j\partial_k f \quad \left(= \varepsilon_{ikj}\partial_k\partial_j f = -\varepsilon_{ijk}\partial_j\partial_k f\right) = 0$$

- $\boxed{\text{grad div}}$
$$[\nabla(\nabla\cdot F)]_i = \partial_i(\partial_j F_j) = \partial_i\partial_j F_j$$

- $\boxed{\text{div curl} = 0}$
$$\nabla\cdot(\nabla\wedge F) = \partial_i(\varepsilon_{ijk}\partial_j F_k) = \varepsilon_{ijk}\partial_i\partial_j F_k = \varepsilon_{kij}\partial_i\partial_j F_k$$
$$= \sum_k k^{th} \text{ coordinate of curl grad } F_k = \sum_k 0 = 0$$

- $\boxed{\begin{array}{c}\text{Curl curl} = \text{grad div} - \text{Laplacian} \\ \nabla\wedge(\nabla\wedge F) = \nabla(\nabla\cdot F) - \nabla^2 F\end{array}}$
$$[\nabla\wedge(\nabla\wedge F)]_i = \varepsilon_{ijk}\partial_j(\nabla\wedge F)_k = \varepsilon_{kij}\partial_j\varepsilon_{klm}\partial_l F_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l F_m = \partial_i\partial_j F_j - \partial_j\partial_j F_i$$
$$= \nabla(\nabla\cdot F) - \nabla\cdot(\nabla F) = \text{grad div } F - \text{Laplacian } F$$

where the Laplacian, ∇^2, is being applied to the entire vector F.

"Extension operators"
grad : $(\mathbb{R}^3\to\mathbb{R}^n) \to (\mathbb{R}^3\to\mathbb{R}^{3\times n})$
div grad (Laplacian) : $(\mathbb{R}^3\to\mathbb{R}^3) \to (\mathbb{R}^3\to\mathbb{R}^3)$

Laplacian & curl commute:
$$[\nabla\wedge\nabla^2 u]_i = \varepsilon_{ijk}\partial_j(\partial_l^2 u_k) = \partial_l^2\varepsilon_{ijk}\partial_j u_k = [\nabla^2(\nabla\wedge u)]_i$$

Triple Vector Product

$a\times(b\times c)_i = \varepsilon_{ijk}\, a_j (b\times c)_k$

$= \varepsilon_{ijk}\, a_j\, \varepsilon_{klm}\, b_l c_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j (b_l c_m)$

$= a_j b_i c_j - a_j b_j c_i$

$= b(a\cdot c) - c(a\cdot b)$

$(a\times b)\times c = -c\times(a\times b)$

$= c\times(b\times a)$

$= b(a\cdot c) - a(c\cdot b)$

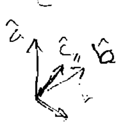
In words:

A cross product cross a vector multiplies the vector in the middle by the scalar product of the vectors on the ends and subtracts the other vector inside the parentheses times the scalar product of the (other two) outside vector with the other vector.

- It is a linear combination of the vectors in parentheses weighted by the projection of the outside vector onto the two inside vectors
- It is perpendicular to the outside vector.
- It is perpendicular to the projection of the outside vector onto the plane spanned by a & b.
- It is rotated from $\hat c$ in the direction from $\hat a$ to $\hat b$.

Pf

$(a\times b)\times c = \left[\hat b\cdot(\hat a\cdot\hat c) - \hat a(\hat b\cdot\hat c)\right] \|a\|\,\|b\|\,\|c\|$.



Again:

$[(a\wedge b)\wedge c]_i = \varepsilon_{ijk}\,(a\wedge b)_j\, c_k$

$= \varepsilon_{ijk}\,\varepsilon_{jlm}(a_l b_m) c_k$

$= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(a_l b_m) c_k$

$= \boxed{(a_k b_i) c_k - (a_i b_k) c_k}$

$= (a\cdot c)\, b - a(b\cdot c)$ (iff assoc. & comm.)

$c\cdot(a\otimes b) - (a\otimes b)\cdot c$

Need to use this form if components are differential operators

And:

$[a\wedge(b\wedge c)]_i = \varepsilon_{ijk}\, a_j (b\wedge c)_k$

$= \varepsilon_{ijk}\, a_j\, \varepsilon_{klm}(b_l c_m)$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\, a_j (b_l c_m)$

$= \boxed{a_j(b_i c_j) - a_j(b_j c_i)}$

$= b(a\cdot c) - c(a\cdot b)$

Application

$(\nabla\wedge u)\wedge u = (\partial_j u_i) u_j - (\partial_i u_j) u_j$

$= (u\cdot\nabla)u - \frac{1}{2}\nabla(u\cdot u)$

Vector identities

$$\boxed{(u\cdot\nabla)u = (\nabla\wedge u)\wedge u + \nabla(\tfrac{1}{2}u^2)}$$

Pf

$$
\begin{aligned}
LHS_i &= (u\cdot\nabla)u_i \\
&= u_k\partial_k u_i \\
RHS_i &= [(\nabla\wedge u)\wedge u]_i + [\nabla(\tfrac12(u\cdot u))]_i \\
&= \varepsilon_{ijk}(\nabla\wedge u)_j u_k + \tfrac12\partial_i(u_j u_j) \\
&= \varepsilon_{jki}(\varepsilon_{jlm}\partial_l u_m)u_k + \tfrac12\cdot 2u_j(\partial_i u_j) \\
&= (\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})(\partial_l u_m)u_k + u_j(\partial_i u_j) \\
&= \delta_{kl}\delta_{im}(\partial_l u_m)u_k - \delta_{km}\delta_{il}(\partial_l u_m)u_k \\
&\quad + u_j(\partial_i u_j) \\
&= (\partial_k u_i)u_k - (\partial_i u_k)u_k + u_k(\partial_i u_k) \\
&= (\partial_k u_i)u_k, \text{ as desired.}
\end{aligned}
$$

Derivatives of products

$$
\begin{aligned}
[\nabla(fg)]_i &= \partial_i(fg) \\
&= f(\partial_i g) + g(\partial_i f) \\
&= [f\nabla g + g\nabla f]_i
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot(fu) &= \partial_i(fu_i) \\
&= f\partial_i u_i + u_i\partial_i f \\
&= [f\nabla\cdot u + u\cdot\nabla f]_i = [\nabla f\cdot u + f\nabla\cdot u]_i
\end{aligned}
$$

$$
\begin{aligned}
[\nabla\wedge(fu)]_i &= \varepsilon_{ijk}\partial_j(fu_k) \\
&= \varepsilon_{ijk}[(\partial_j f)u_k + f(\partial_j u_k)] \\
&= [(\nabla f)\wedge u + f(\nabla\wedge u)]_i
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot(u\wedge v) &= \partial_i(u\wedge v)_i \\
&= \partial_i(\varepsilon_{ijk}u_j v_k) \\
&= \varepsilon_{ijk}[(\partial_i u_j)v_k + (\partial_i v_k)u_j] \\
&= (\varepsilon_{kij}\partial_i u_j)v_k - (\varepsilon_{jik}\partial_i v_k)u_j \\
&= (\nabla\times u)\cdot v - (\nabla\times v)\cdot u
\end{aligned}
$$

$$
\begin{aligned}
[\nabla\times(u\times v)]_i &= \varepsilon_{ijk}\partial_j(u\times v)_k \\
&= \varepsilon_{ijk}\partial_j(\varepsilon_{klm}u_l v_m) \\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})[(\partial_j u_l)v_m + u_l\partial_j v_m] \\
&= (\partial_j u_i)v_j - (\partial_j u_j)v_i + u_i\partial_j v_j - u_j\partial_j v_i \\
&= \boxed{(v\cdot\nabla u - u\cdot\nabla v) + \boxed{u(\nabla\cdot v) - v(\nabla\cdot u)}}
\end{aligned}
$$

where $\boxed{u\cdot\nabla = u_j\partial_j}$

can operate on a vector,

since ∂_j can operate on a vector.

what about composing other operators with this operator?

$$
\begin{aligned}
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\partial_j(u_l v_m) \\
&= \partial_j(u_i v_j) - \partial_j(u_j v_i) \\
&= v_j\partial_j u_i + u_i\partial_j v_j - u_j\partial_j v_i - v_i\partial_j u_j
\end{aligned}
$$

Let $f,g : \mathbb{R}^3 \to \mathbb{R}$

$u,v : \mathbb{R}^3 \to \mathbb{R}^3$

Vector products

			operations
fg :	$\mathbb{R}^3$	$\to \mathbb{R}$	∇
fu :	$\mathbb{R}^3$	$\to \mathbb{R}^3$	$\nabla\cdot$, $\nabla\wedge$
$u\wedge v$:	$\mathbb{R}^3$	$\to \mathbb{R}^3$	$\nabla\cdot$, $\nabla\wedge$
$u\cdot v$:	$\mathbb{R}^3$	$\to \mathbb{R}$	∇

$$\boxed{\nabla(u\cdot v)} = u\wedge(\nabla\wedge v) + v\times(\nabla\times u) + u\cdot\nabla v + v\cdot\nabla u$$

(see Mathews 79)

Identities involving $u\cdot\nabla$

$$\boxed{\nabla\wedge(u\cdot\nabla u) = u\cdot(\nabla\wedge u) + (\nabla\wedge u)\cdot\nabla u} \text{ (how?)}$$

$$\nabla\wedge(u\cdot\nabla w) = ?$$

Take $\nabla\wedge$ of both sides.

$$
\begin{aligned}
\nabla\wedge(u\cdot\nabla u) &= \nabla\wedge((\nabla\wedge u)\wedge u) \\
&= u\cdot\nabla(\nabla\wedge u) - (\nabla\wedge u)\cdot\nabla u \\
&\quad + (\nabla\wedge u)(\nabla\cdot u) - u\underbrace{(\nabla\cdot\nabla\wedge u)}_{0}
\end{aligned}
$$

Again:

$$
\begin{aligned}
(\nabla\wedge u)\wedge w &= \varepsilon_{ijk}\varepsilon_{jlm}(\partial_l u_m)w_k \\
&= (\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})(\partial_l u_m)w_k \\
&= (\partial_k u_i)w_k - (\partial_i u_k)w_k \\
&= w\cdot\nabla u - \nabla u\cdot w
\end{aligned}
$$

So if $w = u$, then

$$
\begin{aligned}
(\nabla\wedge u)\wedge u &= u\cdot\nabla u - (\nabla u)\cdot u \\
&= u\cdot\nabla u - \nabla(\tfrac12 u\cdot u)
\end{aligned}
$$