Vector Calculus

Derivation of Gauss's Law (from FTC)

Show: $\int_V \frac{\partial f}{\partial x_i} = \int_{\partial V} n_i f$ $\quad V \in \mathbb{R}^n$

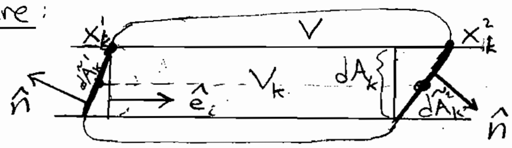
where $n_i = \hat{n}\cdot\hat{e}_i$,

$\hat{n}$ = outward normal,

$\hat{e}_i$ = unit vector in direction of x_i axis.

Idea: Slice V into rods (or cylinders) parallel to x_i axis and apply the Fundamental Theorem of Calculus. WLOG $n \geq 2$.

Picture:



Recall FTC: $\int_a^b \frac{\partial f}{\partial x_i}dx_i = [f]_a^b$

$$\int_V \frac{\partial f}{\partial x_i} = \sum_{k=1}^m \int_{V_k} \frac{\partial f}{\partial x_i}$$

Could write $O(\varepsilon^2)$ if assuming differentiability

$$= \left(\sum_{k=1}^m dA_k \int_{x_k^1}^{x_k^2} \frac{\partial f}{\partial x_i}\right) + m\cdot O(\varepsilon)\cdot d\tilde{A}_{max}$$

[where $\varepsilon = O(\sqrt[n-1]{d\tilde{A}_k})$, $d\tilde{A}_{max} = O(\frac{1}{m})$

(This should handle regions of ∂V where $\hat{n}\cdot\hat{e}_i \approx 0$)

where $d\tilde{A}_{max} = \max(\text{all } d\tilde{A}\text{'s})$]

$$= \sum_{k=1}^m dA_k [f(x_k^2) - f(x_k^1)] + O(m^{\frac{-1}{n-1}})$$

[But $dA_k = d\tilde{A}_k^2\, \hat{n}^2\cdot\hat{e}_i = d\tilde{A}_k^2 n_i^2$

$\quad = -d\tilde{A}_k^1\, \hat{n}^1\cdot\hat{e}_i = -d\tilde{A}_k^1 n_i^1$]

$$= \sum_{k=1}^m d\tilde{A}_k^2 n_i^2 f(x_k^2) + d\tilde{A}_k^1 n_i^1 f(x_k^1) + O(m^{\frac{-1}{n-1}})$$

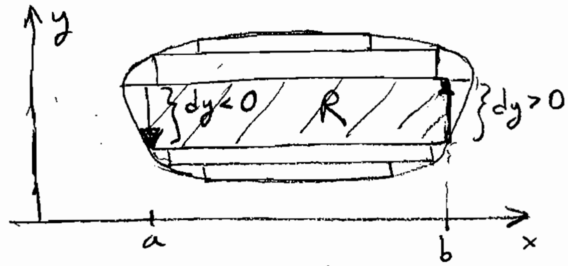
$$= \int_{\partial V} n_i f \quad \text{as } m \to \infty.$$

Green's Theorem in the plane (special cases of Gauss/Stokes)

2 Basic identities:

① $\oint_{\partial A} N(x,y)\,dy = \iint_A \frac{\partial N}{\partial x}(x,y)\,dx\,dy$

Why?:



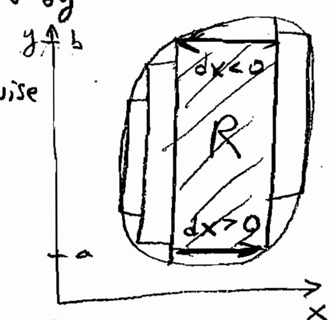
- True for a thin rectangle:

$$\oint_{\partial R} N\,dy \simeq (N(b,y) - N(a,y))dy = \iint_R \frac{\partial N}{\partial x}(x,y)\,dx\,dy$$

- So true for any region in plane.

② $\oint_{\partial A} M(x,y)\,dx = - \iint \frac{\partial M}{\partial y}(x,y)\,dx\,dy$

Why? Same reason: For a counterclockwise path integral, dx is negative on the high side (of y), so a negative sign appears:



$$\oint_{\partial R} M\,dx \simeq [M(x,b) - M(x,a)](-dx) = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

Let $\vec{F}(x,y) = (M(x,y), N(x,y))$. $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

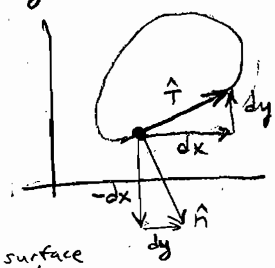
Gauss's divergence theorem:

$$\iint \underline{\nabla}\cdot\underline{F}\,dx\,dy = \iint(\partial_x M + \partial_y N)\,dx\,dy$$

$$= \oint M\,dy - N\,dx$$

$$= \oint \binom{M}{N}\cdot\binom{dy}{-dx}$$

$$= \oint \underline{F}\cdot\hat{n}\,ds$$



Stokes theorem:

$\iint(\nabla\times F)\cdot\hat{k}$ ← normal to surface is in z direction.

$$= \iint(\partial_x N - \partial_y M)\,dx\,dy$$

$$= \oint N\,dy + M\,dx$$

$$= \oint \binom{M}{N}\cdot\binom{dx}{dy}$$

$$= \oint \underline{F}\cdot\hat{T}\,ds$$

Vector Calculus

Stoke's Theorem derived from Gauss

Let A be an "orientable" surface in $\mathbb{R}^3$.
Let $\hat{n}: A \to S^2$ be the unit normal to A toward the side designated as "positive".
Let $\hat{T}: \partial A \to S^2$ be the unit tangent to ∂A directed counterclockwise around ∂A with respect to $\hat{n}$; i.e. so that $\hat{T} \times \hat{n}$ on ∂A points to the exterior of A.

Then Stoke's Theorem claims that

$$\oint_{\partial A} \underline{u} \cdot \hat{T}\, d\ell = \int_A \hat{n} \cdot \nabla \times \underline{u}\, dA \quad \forall \underline{u}: \mathbb{R}^3 \xrightarrow{\text{smooth}} \mathbb{R}^3.$$

Note that others write $\underline{d\ell} \equiv \hat{T}\, d\ell$.

To arrive at this theorem, partition A into small pieces that are approximately linear.
If A is differentiable, then we can approximate A by linear tiles $\tilde{A}_i$ with area $O(\varepsilon)$ and which deviate from A by a maximum of $O(\varepsilon^2)$.
So if the theorem is true for each linear piece, then we will have Stoke's theorem by letting $\varepsilon \to 0$:

$$\int_A \hat{n} \cdot \nabla \times u\, dA = \sum_{i=1}^{n} \int_{A_i} \hat{n} \cdot \nabla \times u\, dA$$

$$= \left(\sum_{i=1}^{m} \int_{\tilde{A}_i} \hat{n} \cdot \nabla \times u\, dA \right) + O(\varepsilon^2) \cdot O(\varepsilon^2) \cdot m$$

[where $\varepsilon = O\left(\frac{\text{radius}(A)}{\sqrt{m}} \right)$]

$$= \left(\sum_{i=1}^{m} \oint_{\partial \tilde{A}_i} \underline{u} \cdot \hat{T}\, d\ell \right) + O(\varepsilon^4)\, m$$

$$= \left(\sum_{i=1}^{m} \oint_{\partial A_i} \underline{u} \cdot \hat{T}\, d\ell \right) + O(\varepsilon^2) \cdot O(\varepsilon) \cdot m + O(\varepsilon^4)\, m$$

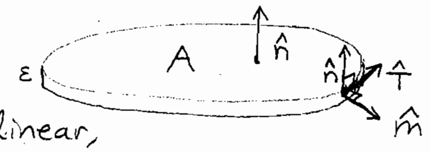
$$= \left(\oint_{\partial A} \underline{u} \cdot \hat{T}\, d\ell \right) + O(\varepsilon^3) \cdot m$$

$$= \oint_{\partial A} \underline{u} \cdot \hat{T}\, d\ell + O(m^{-\frac{1}{2}})$$

$$= \oint_{\partial A} \underline{u} \cdot \hat{T}\, d\ell \quad \text{as } m \to \infty.$$

To prove Stoke's Theorem for a linear surface A, define a volume element V of thickness ε whose "upper surface" is A.
(So $V = A - \varepsilon \hat{n}\, [0,1]$.)

Picture:



Since A is linear, $\hat{n}$ is a constant.
Let $\hat{m}$ denote the outward normal of V.

$$\varepsilon \int_A \hat{n} \cdot \nabla \times u\, dA = \varepsilon \hat{n} \cdot \int_A \nabla \times \underline{u}\, dA$$

$$= \left(\hat{n} \cdot \int_V \nabla \times \underline{u}\, dV \right) + O(\varepsilon^2)$$

[assuming $\underline{u}$ has a bounded 1^{st} derivative.]

$$= \hat{n} \cdot \int_{\partial V} \hat{m} \times \underline{u} \quad + O(\varepsilon^2)$$

$$= \hat{n} \cdot \left(\int_A \hat{n} \times \underline{u} \right) + \hat{n} \cdot \left(\int_{A - \varepsilon \hat{n}} (-\hat{n}) \times \underline{u} \right) + \hat{n} \cdot \left(\int_{\partial A - \varepsilon \hat{n}[0,1]} \hat{m} \times \underline{u} \right) + \underbrace{O(\varepsilon^2) + O(\varepsilon^2)}_{O(\varepsilon^2)}$$

$$\begin{bmatrix} \text{But } \hat{n} \cdot (\hat{n} \times \underline{u}) = \underline{u} \cdot (\hat{n} \times \hat{n}) = 0, \\ \text{and } \hat{n} \cdot (\hat{m} \times \underline{u}) = \underline{u} \cdot (\hat{n} \times \hat{m}) = \underline{u} \cdot \hat{T} \end{bmatrix}$$

$$= \varepsilon \int_{\partial A} \underline{u} \cdot \hat{T}\, d\ell + O(\varepsilon^2)$$

So $\displaystyle\int_A (\nabla \times \underline{u}) \cdot \hat{n}\, dA = \oint_{\partial A} \underline{u} \cdot \hat{T}\, d\ell$,

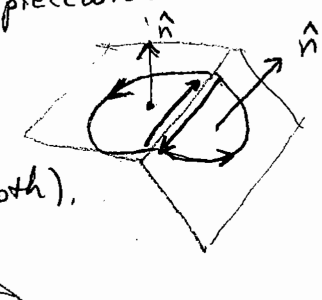
as desired.

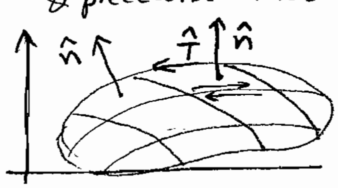
Stoke's Theorem in $\mathbb{R}^3$ derived from Stoke's Theorem in $\mathbb{R}^2$

Claim: $\displaystyle\iint_S (\nabla \times F) \cdot \hat{n} = \oint_{\partial S} F \cdot \hat{T}$

Justification:

- True in $\mathbb{R}^2$: $\left[\nabla \times \begin{pmatrix} M \\ N \\ P \end{pmatrix} \right] \cdot \hat{k} = \partial_x N - \partial_y M$
- So true for any flat surface
- So true for any piecewise flat surface
- So true for any smooth surface (or continuous & piecewise smooth).





Understanding the Laplacian

The key to understanding the physical meaning of any differential operator is to apply Gauss's law (or Stoke's theorem) to the integral of this operator over a region.

Let Ω = infinitesimal ball with radius R.

I claim that the Laplacian evaluated at (the center of) this ball tells you the average value of the function over $\partial\Omega$ (or over Ω) minus the value at the center. I wish to determine the exact nature of this relationship.

$$\text{Let } I = \int_\Omega \nabla^2 u \quad \leftarrow \text{Avg (Laplacian over } \Omega) \cdot \text{Vol}(R)$$

$$= \int_\Omega \nabla \cdot \nabla u$$

$$= \int_{\partial\Omega} n \cdot \nabla u$$

$$= \int_{\partial\Omega} \frac{\partial u}{\partial n}$$

The value of the Laplacian at the center of the sphere is:

$$\nabla^2 u(0) = \lim_{R\to 0} \frac{\int_{\Omega(R)} \nabla^2 (u)}{\text{Vol}(R)} \quad \text{where } \text{Vol}(R) = \tfrac{4}{3}\pi R^3$$

$$= \lim_{R\to 0} \frac{\int_{\partial\Omega} \frac{\partial u}{\partial n}}{\text{Vol}(R)}$$

Recall that $\text{Area}(R) = 4\pi R^2$

So $\frac{\text{Vol}(R)}{\text{Area}(R)} = \frac{1}{3}R = \frac{1}{m}R = \frac{R}{m}$

where m is dim. of space.

I will show that the average value of $u - u(x_0)$ over Ω (or $\partial\Omega$) and $\int_{\partial\Omega} \frac{\partial u}{\partial n}$ can both be expressed in terms of $\int_{\hat{n}} \frac{\partial^2 u}{\partial \hat{n}^2}$.

Assume $u(r\hat{n}) = ar^2 + br + c$.
(coefficients are functions of $\hat{n}$ but not of r.)

Note: $\frac{\partial u}{\partial \hat{n}} = \frac{\partial u(r\hat{n})}{\partial r} = 2ar + b$, $\frac{\partial^2 u}{\partial \hat{n}^2} = 2a$

$u(-r\hat{n}) = ar^2 - br + c$

$\frac{\partial u}{\partial(-\hat{n})} = \frac{\partial u(-r\hat{n})}{\partial r} = 2ar - b$

(1) Now $\int_{\partial\Omega} \frac{\partial u}{\partial n} dS$

$$= \int_{\partial\Omega} 2aR + b \, dS$$

$$= \int_{\partial\Omega^+} (2aR+b) - (2a(-R)+b) \, dS$$

where $\partial\Omega^+$ is a half-sphere (e.g. $x_1 > 0$)

$$= \int_{\partial\Omega^+} 4aR \, dS$$

$$= \int_{\partial\Omega} 2aR \, dS$$

$$= R \int_{\partial\Omega} \frac{\partial^2 u}{\partial n^2} dS$$

$$= R \cdot \text{Avg}\left(\frac{\partial^2 u}{\partial \hat{n}^2} \text{ over all } \hat{n}\right) \cdot \text{Area}(R)$$

But $\int_{\partial\Omega} \frac{\partial u}{\partial n} dS = \text{Avg (Laplacian over } \Omega) \cdot \text{Vol}(R)$

(2) $\text{Avg}(u - u(0) \text{ over } \partial\Omega) \cdot \text{Area}(R)$

$\text{Avg}(u(R\hat{n}) - u(0)) \cdot \text{Area}(R)$

$$= \int_{\partial\Omega} (aR^2 + bR + c) - c \, dS$$

$$= \int_{\partial\Omega^+} (aR^2 + bR) + (a(-R)^2 + b(-R)) \, dS$$

$$= \int_{\partial\Omega^+} 2aR^2 \, dS$$

$$= \int_{\partial\Omega} aR^2 \, dS$$

$$= R^2 \int_{\partial\Omega} \frac{\partial^2 u}{\partial n^2} dS = R^2 \cdot \text{Avg}\left(\frac{\partial^2 u}{\partial \hat{n}^2} \text{ over all } \hat{n}\right) \cdot \text{Area}(R)$$

(3) $\text{Avg}(u \text{ along } \hat{n})$

$$= \frac{\int_{-R}^{R} (ar^2 + \cancel{br} + c)|r|^{m-1} dr}{\int_{-R}^{R} |r|^{m-1} dr} \quad \text{(weighted avg)}$$

$$= \frac{\cancel{2}\int_0^R (ar^{m+1} + cr^{m-1}) \, dr}{\cancel{2}(\frac{1}{m}) R^m}$$

$$= \frac{\frac{a}{m+2} R^{m+2} + \frac{c}{m} R^m}{\frac{1}{m} R^m}$$

$$= \left(\frac{m}{m+2}\right) aR^2 + c$$

So $\text{Avg}(u - u(0) \text{ along } \hat{n})$

$$= \left(\frac{m}{m+2}\right) aR^2 = \left(\frac{m}{m+2}\right)\left(\frac{1}{2}\frac{\partial^2 u}{\partial \hat{n}^2}\right) R^2$$

So $\text{Avg}(u - u(0) \text{ over } \Omega)$

$$= \frac{1}{2}\left(\frac{m}{m+2}\right) R^2 \cdot \text{Avg}\left(\frac{\partial^2 u}{\partial \hat{n}^2} \text{ over all } \hat{n}\right)$$

since $\frac{\partial^2 u}{\partial \hat{n}^2}$ is assumed to be constant along any $\hat{n}$.

So $\text{Avg}(u - u(0) \text{ over } \Omega) = \frac{1}{2}\left(\frac{m}{m+2}\right) \cdot \text{Avg}(u - u(0) \text{ over } \partial\Omega)$

Conclusion

$$\nabla^2 u(0) \approx \text{Avg(Laplacian over } \Omega)$$

$$= \int_{\partial\Omega} \frac{\partial u}{\partial n} dS \,/\, \text{Vol}(R)$$

$$= R \cdot \frac{\text{Area}(R)}{\text{Vol}(R)} \text{Avg}\left(\frac{\partial^2 u}{\partial \hat{n}^2} \text{ over } \hat{n}\right)$$

$$= R \cdot \frac{\text{Area}(R)}{\text{Vol}(R)} \frac{\text{Avg}(u - u(0) \text{ over } \partial\Omega)}{R^2}$$

Note:

$$\frac{R \cdot \text{Area}(R)}{\text{Vol}(R)} = R\left(\frac{m}{R}\right) = m$$

in R^m

$$\text{So } \boxed{\nabla^2 u(0) = m \underbrace{\frac{\text{Avg}(u - u(0) \text{ over } \partial\Omega)}{R^2}}_{\lim \text{ as } R \to 0}}$$